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A CENTRAL LIMIT THEOREM FOR NON-LINEAR FUNCTIONS OF A NORMAL STATIONARY PROCESS

by

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ERRATA

Page	Line	
3	7	Change "$\alpha,\ = 1,2,\ldots,n$" to "$\alpha,\beta=1,2,\ldots,n$".
3	9	Change "$X_{t_1+k_2}(\alpha)$" to "$X_{t+k_2}(\alpha)$".
3	11	Change "positives" to "positive"
5	13	Change " $\{1,2,3\} = \{3,2,1\} = \{1,3,2\} = \{3,1,2\} = \{2,1,3\} = \{2,3,1\}$ "
		to " $\{a_1,a_2,a_3\} = \{a_3,a_2,a_1\} = \{a_1,a_3,a_2\} = \{a_3,a_1,a_2\} = \{a_2,a_1,a_3\} = \{a_2,a_3,a_1\}$".
13	11	Change "$N(\varepsilon)$" to "$N(\varepsilon,\delta)$".
13	14	Change "$\delta(\varepsilon)/2$" to "$\delta/2$".
14	8	Change "if $N > N(\varepsilon)$" to if $\delta < \delta(\varepsilon)$ and $N > N(\varepsilon,\delta)$".
15	14	Change "t_{1_e} " to "t_1".
18	11	Change "$k_{p_1}^{(\beta)}$" to "$k_{p_2}^{(\beta)}$".
19	5	Change "2 lemma" to "2 lemmas".
20	2,3	Change all "$\delta(\varepsilon)$" to "δ".
20	6	Change "if $N > [\frac{C_1}{\sin\frac{1}{2}\delta(\varepsilon)}]^2$ " to "if $\delta < \delta(\varepsilon)$ and $N > [\frac{C_1}{\sin\frac{1}{2}\delta}]^2$".

Page	Line	
25	15	Change "$\cup S_p = \{(s_1,s_2)\cup\cdots\cup(s_{2\ell-1},s_{2\ell})\}$" to "$\cup S_p = (s_1,s_2)\cup\cdots\cup(s_{2\ell-1},s_{2\ell})$".
31	10	Change "$p_\omega < b$" to "$p_\omega < b_\omega$".
34	3	Change "(α_{q_m})" to "$(\alpha_{q_{\ell m}})$".
38	8	Change "$-\lambda_{\ell-1}+u_{\ell,1}-u_{\ell,2}+\cdots+u_{\ell,m-1}$" to "$-\lambda_{\ell-1}+u_{\ell,1}+u_{\ell,2}+\cdots+u_{\ell,m-1}$".
42	10	Add "$\}$" at the end of this line.
51	20	Change "lemma 3.1" to "lemma 3.2"
53	15	Change "the limit r" to "this limit".

A Central Limit Theorem for Non-linear Functions of a Normal Stationary Process*

by

Tze-Chien Sun

Introduction.

Let X_t, $t = 0, \pm 1, \pm 2, \cdots$ be a real normal stationary process with mean $EX_t \equiv 0$ for all t and covariances

$$r_k = \mathrm{cov}(X_j, X_{j+k}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda \ , \ k=0, \pm 1, \pm 2, \cdots$$

where $f \in L^2(-\pi, \pi)$. Since X_t is a real process, $f(\lambda)$ is symmetrical with respect to $\lambda=0$, i.e. $f(-\lambda) = f(\lambda)$ for $-\pi \leq \lambda \leq \pi$. Since r_k is a positive definite sequence, $f(\lambda) \geq 0$ for $-\pi \leq \lambda \leq \pi$.

Let

$$\begin{aligned} c_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda) d\lambda \leq C_1 = \max(1, c_1) \\ c_2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f^2(\lambda) d\lambda \leq C_2 = \max(1, c_2). \end{aligned} \tag{1}$$

Note that $f(\lambda)$ above is defined only on $[-\pi, \pi]$. However we may at times find it necessary for later use to extend $f(\lambda)$ to the whole real line by periodicity with period 2π, i.e.

$$f(\lambda + 2n\pi) = f(\lambda) \ , \ -\pi \leq \lambda < \pi \ , \ n=0, \pm 1, \pm 2, \cdots .$$

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It has been known that a stationary process generated by any linear function of a normal stationary process is again normal if it exists in the mean. But a stationary process generated by a non-linear function of a normal stationary process is not in general normal, e.g. X_t^2 has as its marginal distribution a chi-square distribution and hence is not normal. However, it is a very interesting question to ask what conditions we should impose on the original process and the form of the non-linear functions so that the new stationary processes generated by those non-linear functions of a normal stationary process obey the Central Limit Theorem. The solution of this problem will provide more knowledge on the non-linear problems and, in particular, will help us to carry out estimations of parameters involving non-linear functions of the process.

It was proved by Rosenblatt [1], estimating the maximum eigenvalue of a Toeplitz matrix that if

$r_k^* = \frac{1}{N} \sum_{j=1}^{N-|k|} X_j X_{j+k}$, $k=0,1,\ldots,s$, are the covariance estimates of r_k, and $f \in L^2$, then

$$\sqrt{N}(r_k^* - Er_k^*) \ , \ k=0,1,\ldots,s$$

are asymptotically jointly normally distributed with mean zero and covariances

$$c_{j,k} = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos j\lambda \cos k\lambda \, f^2(\lambda) d\lambda \ , \ j,k=1,2,\ldots,s.$$

The result obtained here is a generalization of that given above.

We shall first prove under the conditions

$$\text{(i)} \quad f \in L^2 \qquad \text{(ii)} \quad \lim_{N\to\infty} \frac{1}{2\pi N} \int_{-\pi}^{\pi} \frac{\sin^2 \frac{N}{2}\lambda}{\sin^2 \frac{1}{2}\lambda} f(\lambda)d\lambda \text{ exists and}$$

is finite, for the case m any positive odd integer, that

$$\frac{1}{N^{\frac{1}{2}}} \sum_{t=1}^{N} X_{t+k_1^{(\alpha)}} X_{t+k_2^{(\alpha)}} \cdots X_{t+k_m^{(\alpha)}}$$

$\alpha=1,2,\ldots,n$ where n is a positive integer and $k_1^{(\alpha)}, k_2^{(\alpha)}, \cdots, k_m^{(\alpha)}$ are integers, are asymptotically, jointly normally distributed with mean zero and certain finite covariances $r_{\alpha,\beta}$, α, $=1,2,\ldots,n$. The main result will be that under conditions (i) and (ii)

$$\frac{1}{N^{\frac{1}{2}}} \sum_{t=1}^{N} (X_{t+k_1^{(\alpha)}} X_{t_1+k_2^{(\alpha)}} \cdots\cdots X_{t+k_{m_\alpha}^{(\alpha)}}$$

$$- EX_{t+k_1^{(\alpha)}} X_{t+k_2^{(\alpha)}} \cdots\cdots X_{t+k_{m_\alpha}^{(\alpha)}})$$

$\alpha=1,2,\ldots,n$ where n and m_α are positives integers and $k_1^{(\alpha)},\ldots,k_{m_\alpha}^{(\alpha)}$ are integers, are asymptotically, jointly normally distributed with mean zero and certain finite covariances. However if m_α, $\alpha=1,2,\ldots,n$ are all even, then the condition (ii) is not needed.

(i) and (ii) are sufficient conditions. Whether they are necessary is still unknown. In an example in §2 of [2], it was shown that given any p with $1 < p < 2$ one can construct an $f \in L^p$ but not in L^2 such that one does not have asymptotic normality of the covariance estimate. Because we assumed $f \in L^2$

and so adapted $N^{\frac{1}{2}}$ as the weighing factor and because (ii) is the necessary and sufficient condition for $\frac{1}{N^{\frac{1}{2}}} \sum_{t=1}^{N} X_t$ being asymptotically normal, we see from the above mentioned example that (i) and (ii) are rather reasonable sufficient conditions. Nevertheless, whether every normal stationary process with $f \notin L^2$ fails to have asymptotic normality of the covariance estimate or of any other non-linear functions of the process is still an interesting open problem.

Let

$$Y(t) = \sum_{m=1}^{\infty} \sum_{\substack{k_i=-\infty \\ i=1,\ldots,m \\ k_{i-1} \le k_i}}^{\infty} a_{k_1,\ldots,k_m} (X_{t+k_1} X_{t+k_2} \cdots X_{t+k_m} - EX_{t+k_1} \cdots X_{t+k_m}).$$

A sufficient condition is given for the new process $Y(t)$ having asymptotic normality.

We think it worthwhile to mention some other work on non-linear functions of a random process, e.g. [3], [4], [5], [6], although they are not directly related to our work.

1. An Auxiliary Theorem.

We say $\{a_1, a_2, \ldots, a_n\}$ is an unordered set if $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are considered the same whenever the b_j's are just a permutation of the a_j's. And we say $\{a_1, a_2, \ldots, a_n\}$ is an ordered set if $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are considered different from each other unless $a_j = b_j$ for all $j = 1, 2, \ldots, n$. For example, if $\{a_1, a_2, a_3\}$ represents a 3-dimensional vector, it is an ordered set for $\{a_1, a_2, a_3\} = \{b_1, b_2, b_3\}$, implies $a_1 = b_1, a_2 = b_2, a_3 = b_3$ and if $\{a_1, a_2, a_3\}$ are the result of picking successively 3 numbers from $1, 2, \ldots, n$ and we do not care which was picked out first and which was second and which was third, in this case $\{a_1, a_2, a_3\}$ is an unordered set because $\{1,2,3\} = \{3,2,1\} = \{1,3,2\} = \{3,1,2\} = \{2,1,3\} = \{2,3,1\}$.

We shall call a partition into pairs of a set $A = \{a_1, \ldots, a_n\}$ where n is even, and all a_j are distinct, an unordered collection of mutually disjoint, unordered pairs $A_p = \{a_{j_{2p-1}}, a_{j_{2p}}\}$, $p = 1, 2, \ldots, n/2$, and $j_1, j_2, \ldots, j_n = 1, 2, \ldots, n$ such that $\bigcup_{p=1}^{n/2} A_p = A$.

Theorem 1. Let $T_k = \{t_1, t_2, \ldots, t_k\}$ be a set of k integers. Then

$$EX_{t_1} X_{t_2} \cdots X_{t_k}$$

$$= \begin{cases} 0 & \text{if } k \text{ is odd} \\ \tilde{\sum\limits_{T_k}} r_{t_{j_1} - t_{j_2}} \, r_{t_{j_3} - t_{j_4}} \cdots r_{t_{j_{k-1}} - t_{j_k}} & \text{if } k \text{ is even} \end{cases}$$

where $\tilde{\Sigma}_{T_k}$ denotes summation over all possible partitions into pairs of the set T_k with the understanding that in counting possible partitions into pairs of T_k we shall regard the t_j's as entries distinct from each other instead of taking them as numerical numbers, so that there are exactly $\frac{k!}{2^{k/2}(\frac{k}{2})!}$ terms in $\tilde{\Sigma}_{T_k}$.

[Proof]. Differentiating the multi-variate characteristic function of $X_{t_1}, X_{t_2}, \dots, X_{t_k}$,

$$\varphi(\mu_1, \mu_2, \dots, \mu_k) = E\left\{e^{i\sum_{j=1}^{k}\mu_j X_{t_j}}\right\}$$

$$= e^{-\frac{1}{2}\sum_{j,\ell=1}^{k}\mu_j\mu_\ell r_{t_j - t_\ell}}$$

We shall obtain the following

$$EX_{t_1}X_{t_2}\cdots X_{t_k}$$

$$= \frac{1}{i^k}\left[\frac{\partial^k}{\partial\mu_1\partial\mu_2\cdots\partial\mu_k}\varphi(\mu_1, \mu_2, \dots, \mu_k)\right]_{\mu_1=0,\dots,\mu_k=0}$$

$$= \begin{cases} 0 & \text{if } k \text{ is odd} \\ \dfrac{1}{i^k}(-1)^{k/2}\left[\dfrac{1}{2^{k/2}}\dfrac{1}{(\frac{k}{2})!}\displaystyle\sum_{\substack{\text{over all ordered} \\ \text{collections of mutually} \\ \text{disjoint, ordered pairs} \\ [(t_{j_1}, t_{j_2}), \dots (t_{j_{k-1}}, t_{j_k})] \text{ such that} \\ (t_{j_1}, t_{j_2}) \cup \dots \cup (t_{j_{k-1}}, t_{j_k}) = T_k}} r_{t_{j_1}-t_{j_2}} r_{t_{j_3}-t_{j_4}} \cdots r_{t_{j_{k-1}}-t_{j_k}}\right] & \\ = \tilde{\Sigma}_{T_k} r_{t_{j_1}-t_{j_2}} r_{t_{j_3}-t_{j_4}} \cdots r_{t_{j_{k-1}}-t_{j_k}} & \text{if } k \text{ is even.} \end{cases}$$

Q.E.D.

Suppose we let $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_n\}$ and

$$Y_a = X_{a_1} X_{a_2} \ldots X_{a_m} - EX_{a_1} X_{a_2} \ldots X_{a_m}$$
$$Y_b = X_{b_1} X_{b_2} \ldots X_{b_n} - EX_{b_1} X_{b_2} \ldots X_{b_n}$$

where a_j, $j=1,2,\ldots,m$ and b_j, $j=1,2,\ldots,n$ are any integers. We like to have an explicit formula for

$$EY_a Y_b = E(X_{a_1} X_{a_2} \ldots X_{a_m})(X_{b_1} X_{b_2} \ldots X_{b_n})$$
$$-E(X_{a_1} X_{a_2} \ldots X_{a_m}) E(X_{b_1} X_{b_2} \ldots X_{b_n})$$

in terms of summation over the set $A \cup B$. Note that in taking the union of A and B, we consider them as sets consisting of distinct elements $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_n$ instead of taking them as numerical values.

Corollary 1.1. Let $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_n\}$ and $\beta = \frac{n-m}{2}$. Then

$$EY_a Y_b =$$
$$\begin{cases} 0 & \text{if } m+n \text{ is odd (a)} \\ \sum^*_{A \cup B} r_{a_{j_1} - a_{j_2}} r_{a_{j_3} - a_{j_4}} \cdots r_{a_{j_{2\omega-1}} - a_{j_{2\omega}}} \\ \quad r_{b_{p_1} - b_{p_2}} r_{b_{p_3} - b_{p_4}} \cdots r_{b_{p_{2\omega+2\beta-1}} - b_{p_{2\omega+2\beta}}} \\ \quad r_{a_{j_{2\omega+1}} - b_{p_{2\omega+2\beta+1}}} r_{a_{j_{2\omega+2}} - b_{p_{2\omega+2\beta+2}}} \cdots r_{a_{j_m} - b_{p_n}} \\ & \text{if } m+n \text{ is even (b)} \end{cases}$$

Where $\sum\limits_{A\cup B}^{*}$ denotes all partitions into pairs

$$\{(a_{j_1}a_{j_2}),\ldots,(a_{j_{2\omega-1}},a_{j_{2\omega}}),\ (b_{p_1},b_{p_2}),\ldots(b_{p_{2\omega+2\beta-1}},b_{p_{2\omega+2\beta}}),$$
$$(a_{j_{2\omega+1}},b_{p_{2\omega+2\beta+1}}),\ldots,(a_{j_m},b_{p_n})\}$$

of $A\cup B$ such that $0 \leq 2\omega < m$ and $0 \leq 2\omega+2\beta < n$. Note that in counting the partitions we consider A,B as sets consisting of distinct elements.

[Proof].

(a) is clear.

(b) $E(X_{a_1}\ldots X_{a_m})(X_{b_1}\ldots X_{b_n})$

$$= \tilde{\sum_{A\cup B}} r_{a_{j_1}-a_{j_2}}\ldots r_{a_{j_{2\omega-1}}-a_{j_{2\omega}}}$$
$$r_{b_{p_1}-b_{p_2}}\ldots r_{b_{p_{2\omega+2\beta-1}}-b_{p_{2\omega+2\beta}}}$$
$$r_{a_{j_{2\omega+1}}-b_{p_{2\omega+2\beta+1}}}\ldots r_{a_{j_m}-b_{p_n}}$$

where $0 \leq 2\omega \leq m$, $0 \leq 2\omega+2\beta \leq n$ by Theorem 1. However, those terms in $\tilde{\sum\limits_{A\cup B}}$ with $2\omega=m$ and $2\omega+2\beta=n$ are canceled by $-E(X_{a_1}\ldots X_{a_m})E(X_{b_1}\ldots X_{b_n})$. Hence (b) is established.

2. A Central Limit Theorem for Odd Products.

In Theorem 2, we shall assume in addition to $f \in L^2(-\pi,\pi)$, that

$$\lim_{N\to\infty} \frac{1}{2\pi N} \int_{-\pi}^{\pi} \frac{\sin^2 \frac{N}{2}\lambda}{\sin^2 \frac{1}{2}\lambda} f(\lambda)d\lambda \tag{2}$$

exists and is finite. For convenience of writing, let us define

$$f(0) = \lim_{N\to\infty} \frac{1}{2\pi N} \int_{-\pi}^{\pi} \frac{\sin^2 \frac{N}{2}\lambda}{\sin^2 \frac{1}{2}\lambda} f(\lambda)d\lambda$$

so from here on whenever condition (2) is assumed, we define $f(0)$ as in the above formula. Since the integral

$$\frac{1}{2\pi N} \int_{-\pi}^{\pi} \frac{\sin^2 \frac{N}{2}\lambda}{\sin^2 \frac{1}{2}\lambda} f(\lambda)d\lambda$$

is finite for each N and its limit as $N \to \infty$ exists and is finite, there exist a C_3, $1 < C_3 < \infty$ such that

$$\frac{1}{2\pi N} \int_{-\pi}^{\pi} \frac{\sin^2 \frac{N}{2}\lambda}{\sin^2 \frac{1}{2}\lambda} f(\lambda)d\lambda < C_3 \tag{3}$$

for all N.

We also need the following results:

(i) $$\frac{1}{(2\pi)^2 N} \int_{-\pi}^{\pi}\int_{\pi}^{\pi} \frac{\sin^2 \frac{N}{2}(\lambda_1+\lambda_2+a)}{\sin^2 \frac{1}{2}(\lambda_1+\lambda_2+a)} f(\lambda_1)d\lambda_1 d\lambda_2 = c_1 \le C_1$$

(ii) $$\frac{1}{(2\pi)^2 N} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\sin^2 \frac{N}{2}(\lambda_1+\lambda_2+a)}{\sin^2 \frac{1}{2}(\lambda_1+\lambda_2+a)} f^2(\lambda_1)d\lambda_1 d\lambda_2 = c_2 \le C_2$$

(iii) $$\frac{1}{(2\pi)^2 N} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\sin^2 \frac{N}{2}(\lambda_1+\lambda_2+a)}{\sin^2 \frac{1}{2}(\lambda_1+\lambda_2+a)} f(\lambda_1)f(\lambda_2)d\lambda_1 d\lambda_2 \le C_2 \tag{4}$$

for any real a and for all N. We need to introduce some useful notation here.

Notation 1.

Let m and n be positive integers such that m+n is even. Let $\alpha = \frac{m+n}{2}$ and $\beta = \frac{n-m}{2}$ and $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$ Then, we write

$$\exp\begin{bmatrix} \begin{pmatrix} a_1, a_2, \ldots, a_m \\ b_1, b_2, \ldots, b_n \end{pmatrix} \\ \lambda_1, \lambda_2, \ldots, \lambda_\alpha \end{bmatrix}$$

$$= \sum_{A\cup B}^{*} e^{i(a_{j_1}-a_{j_2})\lambda_1+i(a_{j_3}-a_{j_4})\lambda_2+\ldots+i(a_{j_{2\omega-1}}-a_{j_{2\omega}})\lambda_\omega}$$

$$e^{i(b_{p_1}-b_{p_2})\lambda_{\omega+1}+i(b_{p_3}-b_{p_4})\lambda_{\omega+2}+\ldots+i(b_{p_{2\omega-1+2\beta}}-b_{p_{2\omega+2\beta}})\lambda_{2\omega+\beta}}$$

$$e^{i(a_{j_{2\omega+1}}-b_{p_{2\omega+2\beta+1}})\lambda_{2\omega+\beta+1}+i(a_{j_{2\omega+2}}-b_{p_{2\omega+2\beta+2}})\lambda_{2\omega+\beta+2}} \cdots\cdots$$

$$e^{i(a_{j_m}-b_{p_n})\lambda_\alpha} \cdot \delta(\lambda_{2\omega+\beta+1}+\lambda_{2\omega+\beta+2}+\ldots+\lambda_\alpha, 0)$$

where (i) $\sum_{A\cup B}^{*}$ was defined in Corollary 1.1.

(ii) $\delta(x,0)$ is a δ-function at x=0 such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(x,0)dx = 1. \quad (5)$$

Notation 2. Let

$$K_N(\lambda) = \left|\frac{\sin\frac{N}{2}\lambda}{\sin\frac{1}{2}\lambda}\right| \quad (6)$$

<u>Theorem 2.</u> Suppose $f \in L^2(-\pi,\pi)$ and

$$\lim_{N\to\infty} \frac{1}{2\pi N} \int_{-\pi}^{\pi} K_N^2(\lambda) f(\lambda) d\lambda$$

exists and is finite. Let m be a positive odd integer. Then

$$Y_{N,\alpha} = \frac{1}{N^{\frac{1}{2}}} \sum_{t=1}^{N} X_{t+k_1^{(\alpha)}} X_{t+k_2^{(\alpha)}} \cdots X_{t+k_m^{(\alpha)}}$$

$\alpha=1,2,\ldots,n$ where n is a positive integer and $k_1^{(\alpha)}, k_2^{(\alpha)}, \ldots, k_m^{(\alpha)}$ are integers, are asymptotically, jointly normally distributed with mean zero and covariances

$$r_{\alpha,\beta} = \frac{1}{(2\pi)^m} \int_{-\pi}^{\pi}\cdots\int \exp\begin{bmatrix} \begin{pmatrix} k_1^{(\alpha)}, k_2^{(\alpha)}, \ldots, k_m^{(\alpha)} \\ k_1^{(\beta)}, k_2^{(\beta)}, \ldots, k_m^{(\beta)} \end{pmatrix} \\ \lambda_1 \quad ,\lambda_2 \quad ,\ldots,\lambda_m \end{bmatrix} f(\lambda_1)f(\lambda_2)\ldots f(\lambda_m) \, d\lambda_1 d\lambda_2 \ldots d\lambda_m$$

$\alpha,\beta = 1,2,\ldots,n.$

The proof of Theorem 2 will be given in several steps formulated in terms of several lemmas. In the first two lemmas, we shall show

$$EY_{N,\alpha}Y_{N,\beta} \longrightarrow r_{\alpha,\beta} \qquad \text{as } N \longrightarrow \infty$$

$\alpha,\beta = 1,2,\ldots,n.$

Let $Y_N = \sum_{\alpha=1}^{n} \mu_\alpha Y_{N,\alpha}$ be any linear combination of $Y_{N,\alpha}$.

It will be shown in the remaining lemmas that

$$E\, Y_N^{\ell} \longrightarrow \text{the } \ell^{\text{th}} \text{ moments of a Gaussian distribution}$$
$$\text{with variance } \left(\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \mu_\alpha \mu_\beta r_{\alpha,\beta}\right)$$

The "moment convergence theorem" [7] then assures that Y_N is asymptotically normally distributed. This is just the assertion of Theorem 2.

Lemma 2.1 Suppose $F \in L^2(-\pi,\pi)$ and a is any real number. Then

$$\frac{1}{(2\pi)^2 N} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_N^2(\lambda_1+\lambda_2+a)\,|f(\lambda_1)-f(-\lambda_2-a)|^2 d\lambda_1 d\lambda_2$$

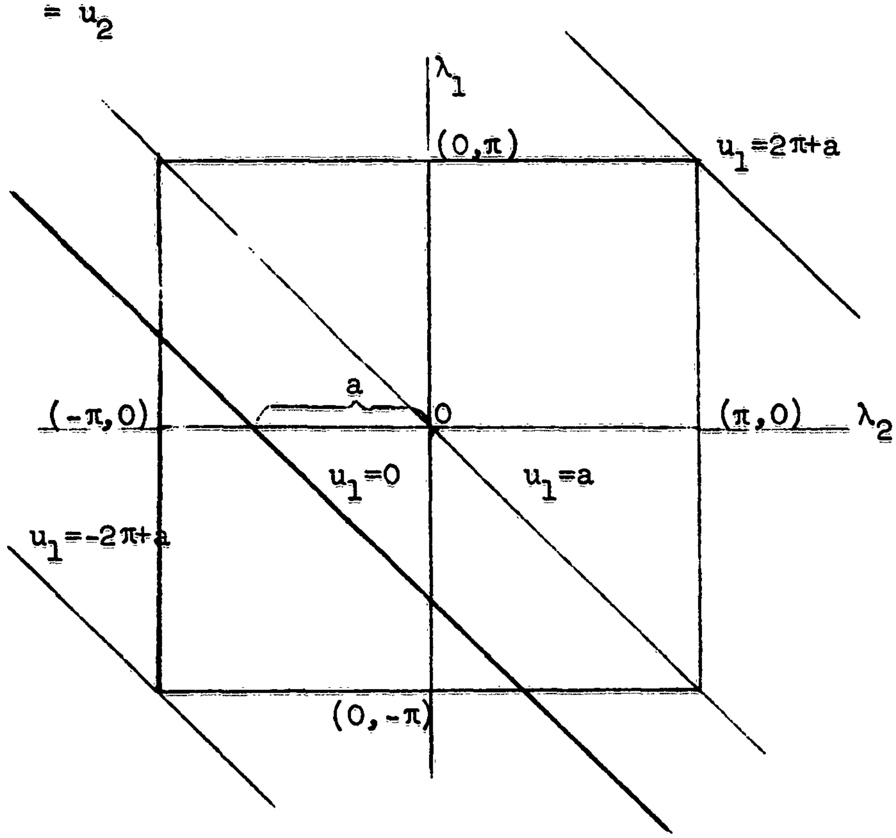
$$\longrightarrow 0$$

uniformly for all a as $N \Longrightarrow \infty$.

[Proof]. First we make the following transformation. Let

$$\lambda_1+\lambda_2+a = u_1$$

$$\lambda_2 = u_2$$



Then, we have

$$\frac{1}{(2\pi)^2 N} \int_{-\pi}^{\pi} \int_{-\pi+u_2+a}^{\pi+u_2+a} K_N^2(u_1) |f(u_1-u_2-a)-f(-u_2-a)|^2 du_1 du_2$$

$$= \frac{1}{(2\pi)^2 N} \int_{a}^{2\pi+a} K_N^2(u_1) \int_{-\pi+u_1-a}^{\pi} |f(u_1-u_2-a)-f(-u_2-a_2)|^2 du_2 du_1$$

$$+ \frac{1}{(2\pi)^2 N} \int_{-2\pi+a}^{a} K_N^2(u_1) \int_{-\pi}^{\pi+u_1-a} |f(u_1-u_2-a)-f(-u_2-a)|^2 du_2 du_1 \qquad (7)$$

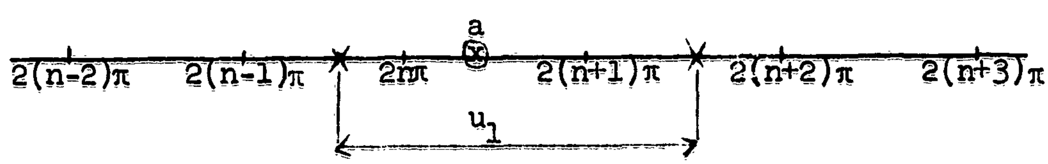
We can rearrange the order of integration because the above integral is absolutely integrable for each N. Since $f \in L^2(-\pi,\pi)$ and f is periodic on the real line with period 2π, for each $\varepsilon > 0$, we can find a $\delta(\varepsilon) > 0$, such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x+\lambda)-f(\lambda)|^2 d\lambda < \frac{\varepsilon}{4}$$

whenever $|x| < \delta(\varepsilon)$. Also let

$$N(\varepsilon) = \text{Integral part of } \left[\frac{4C_2}{\sin^2 \frac{1}{2}\delta} \frac{\varepsilon}{4} + 1\right].$$

Suppose $2n\pi \le a < 2(n+1)\pi$ for some $n=0,\pm1,\pm2,\ldots$, and let



$$S = \{u_1 : |u_1-2k\pi| < \delta(\varepsilon)/2 \text{ for } k = (n-1),\ n,\ (n+1) \text{ or } (n+2)\}$$

$$I_1 = \{u_1: \ a \le u_1 < 2\pi+a\}$$
$$I_2 = \{u_1: \ -2\pi+a \le u_1 < a\}$$

Then (7) becomes

$$\frac{1}{(2\pi)^2 N}\left(\int_{I_1\cap S} + \int_{I_1-S}\right) K_N^2(u_1) \int_{-\pi+u_1-a}^{\pi} |f(u_1-u_2-a)-f(-u_2-a)|^2 du_2 du_1$$

$$+ \frac{1}{(2\pi)^2 N}\left(\int_{I_2\cap S} + \int_{I_2-S}\right) K_N^2(u_1) \int_{-\pi}^{\pi+u_1-a} |f(u_1-u_2-a)-f(-u_2-a)|^2 du_2 du_1$$

$$\le \frac{1}{2\pi N}\int_{-\pi}^{\pi} K_N^2(u_1)\,\frac{\varepsilon}{4}\,du_1 + \frac{1}{2\pi N}\int_{-\pi}^{\pi} \frac{1}{\sin^2 \frac{1}{2}\delta}\, 4\, C_2\, du_1$$

$$+ \frac{1}{2\pi N}\int_{-\pi}^{\pi} K_N^2(u_1)\,\frac{\varepsilon}{4}\,du_1 + \frac{1}{2\pi N}\int_{-\pi}^{\pi} \frac{1}{\sin^2 \frac{1}{2}\delta}\, 4\, C_2\, du_1$$

$$< \varepsilon \qquad\qquad \text{if } N > N(\varepsilon).$$

Hence lemma (2.1) is proved.

Lemma 2.2 Let m be a positive odd integer. Then

$$E\left\{\frac{1}{N}\sum_{t_1=1}^{N}\sum_{t_2=1}^{N}\left(X_{t_1+k_1^{(\alpha)}}X_{t_1+k_2^{(\alpha)}}\cdots X_{t_1+k_m^{(\alpha)}}\right)\left(X_{t_2+k_1^{(\beta)}}X_{t_2+k_2^{(\beta)}}\cdots X_{t_2+k_m^{(\beta)}}\right)\right\}$$

$$\longrightarrow r_{\alpha,\beta} = \frac{1}{(2\pi)^m}\int_{-\pi}^{\pi}\!\!\cdots\!\int \exp\begin{bmatrix} k_1^{(\alpha)}, k_2^{(\alpha)}, \ldots, k_m^{(\alpha)} \\ k_1^{(\beta)}, k_2^{(\beta)}, \ldots, k_m^{(\beta)} \\ \lambda_1, \lambda_2, \ldots, \lambda_m \end{bmatrix} f(\lambda_1)f(\lambda_2)\cdots f(\lambda_m)\, d\lambda_1 d\lambda_2\cdots d\lambda_m$$

as $N \to \infty$ for $\alpha,\beta=1,2,\ldots,n$ and moreover $|r_{\alpha,\beta}| < \infty$.

[Proof]. Let $A_\alpha = \{k_1^{(\alpha)}, \ldots, k_m^{(\alpha)}\}$, $A_\beta = \{k_1^{(\beta)}, \ldots, k_m^{(\beta)}\}$.

Then

$$E\left\{\frac{1}{N} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} (X_{t_1+k_1^{(\alpha)}} X_{t_1+k_2^{(\alpha)}} \cdots X_{t_1+k_m^{(\alpha)}}) (X_{t_2+k_1^{(\beta)}} X_{t_2+k_2^{(\beta)}} \cdots X_{t_2+k_m^{(\beta)}})\right\}$$

$$= \frac{1}{N} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} E\left\{(X_{t_1+k_1^{(\alpha)}} X_{t_1+k_2^{(\alpha)}} \cdots X_{t_1+k_m^{(\alpha)}}) (X_{t_2+k_1^{(\beta)}} X_{t_2+k_2^{(\beta)}} \cdots X_{t_2+k_m^{(\beta)}})\right\}$$

$$= \frac{1}{N} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} \sum^{*}_{A_\alpha \cup A_\beta} r_{(t_1+k_{j_1}^{(\alpha)}-t_1-k_{j_2}^{(\alpha)})} \cdots r_{(t_1+k_{j_{2\omega-1}}^{(\alpha)}-t_1-k_{j_{2\omega}}^{(\alpha)})}$$
$$r_{(t_2+k_{p_1}^{(\beta)}-t_2-k_{p_2}^{(\beta)})} \cdots r_{(t_2+k_{p_{2\omega-1}}^{(\beta)}-t_2-k_{p_{2\omega}}^{(\beta)})}$$
$$r_{(t_1+k_{j_{2\omega+1}}^{(\alpha)}-t_2-k_{p_{2\omega+1}}^{(\beta)})} \cdots r_{(t_1+k_{j_m}^{(\alpha)}-t_2-k_{p_m}^{(\beta)})}$$

where $\sum^{*}_{A_\alpha \cup A_\beta}$ was defined in Corollary (1.1).

$$= \sum^{*}_{A_\alpha \cup A_\beta} \frac{1}{(2\pi)^m N} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N}$$
$$\int_{-\pi}^{\pi}\cdots\int e^{i(k_{j_1}^{(\alpha)}-k_{j_2}^{(\alpha)})\lambda_1+ \cdots +i(k_{j_{2\omega-1}}^{(\alpha)}-k_{j_{2\omega}}^{(\alpha)})\lambda_\omega}$$
$$e^{i(k_{p_1}^{(\beta)}-k_{p_2}^{(\beta)})\lambda_{\omega+1}+\cdots+i(k_{p_{2\omega-1}}^{(\beta)}-k_{p_{2\omega}}^{(\beta)})\lambda_{2\omega}}$$
$$e^{i(k_{j_{2\omega+1}}^{(\alpha)}-k_{p_{2\omega+1}}^{(\alpha)})\lambda_{2\omega+1}+\cdots+i(k_{j_m}^{(\alpha)}-k_{p_m}^{(\alpha)})\lambda_m}$$
$$e^{i(\lambda_{2\omega+1}+\lambda_{2\omega+2}+\cdots+\lambda_m)t_1} e^{-i(\lambda_{2\omega+1}+\lambda_{2\omega+2}+\cdots+\lambda_m)t_2}$$
$$f(\lambda_1)f(\lambda_2)\cdots f(\lambda_m)\,d\lambda_1 d\lambda_2\cdots d\lambda_m$$

$$= \sum^{*}_{A_\alpha \cup A_\beta} \frac{1}{(2\pi)^{m}N} \int_{-\pi}^{\pi}\cdots\int$$

$$e^{i(k^{(\alpha)}_{j_1}-k^{(\alpha)}_{j_2})\lambda_1+\ldots+i(k^{(\alpha)}_{j_{2\omega-1}}-k^{(\alpha)}_{j_{2\omega}})\lambda_\omega}\cdot$$

$$e^{i(k^{(\beta)}_{p_1}-k^{(\beta)}_{p_2})\lambda_{\omega+1}+\ldots+i(k^{(\beta)}_{p_{2\omega-1}}-k^{(\beta)}_{p_{2\omega}})\lambda_{2\omega}}$$

$$\cdot e^{i(k^{(\alpha)}_{j_{2\omega+1}}-k^{(\beta)}_{p_{2\omega+1}})\lambda_{2\omega+1}+\ldots+i(k^{(\alpha)}_{j_m}-k^{(\beta)}_{p_m})\lambda_m}$$

$$K^2_N(\lambda_{2\omega+1}+\lambda_{2\omega+2}+\ldots+\lambda_m)f(\lambda_1)f(\lambda_2)\ldots f(\lambda_m)d\lambda_1 d\lambda_2\ldots d\lambda_m \tag{8}$$

We can change the order of integration because those integrals in (8) are absolutely integrable for each N. Note also that the above integrals are all real. Their imaginary parts are zero for all N because $f(\lambda)$ is symmetric about $\lambda=0$. In order to show that, for $2\omega < m-1$,

$$\frac{1}{(2\pi)^m}\frac{1}{N}\int_{-\pi}^{\pi}\cdots\int e^{i(k^{(\alpha)}_{j_1}-k^{(\alpha)}_{j_2})\lambda_1+\ldots+i(k^{(\alpha)}_{j_m}-k^{(\beta)}_{p_m})\lambda_m}$$

$$K^2_N(\lambda_{2\omega+1}+\lambda_{2\omega+2}+\ldots+\lambda_m)f(\lambda_1)f(\lambda_2)\ldots f(\lambda_m)d\lambda_1 d\lambda_2\ldots d\lambda_m$$

$$\longrightarrow \frac{1}{(2\pi)^{m-1}}\int_{-\pi}^{\pi}\cdots\int e^{i(k^{(\alpha)}_{j_1}-k^{(\alpha)}_{j_2})\lambda_1+\ldots+i(k^{(\alpha)}_{j_m}-k^{(\beta)}_{p_m})(-\lambda_{2\omega+1}-\ldots-\lambda_{m-1})}$$

$$f(\lambda_1)f(\lambda_2)\ldots f(\lambda_{m-1})f(-\lambda_{2\omega+1}-\lambda_{2\omega+2}-\ldots-\lambda_{m-1})$$

$$d\lambda_1 d\lambda_2\ldots d\lambda_{m-1} \tag{9}$$

it suffices to prove

$$\frac{1}{(2\pi)^m} \int_{-\pi}^{\pi}\cdots\int \frac{1}{N} K_N^2(\lambda_{2\omega+1}+\lambda_{2\omega+2}+\cdots+\lambda_m)$$

$$|f(\lambda_1)f(\lambda_2)\cdots f(\lambda_m)-f(\lambda_1)f(\lambda_2)\cdots f(\lambda_{m-1})f(-\lambda_{2\omega+1}-\lambda_{2\omega+2}-\cdots-\lambda_{m-1})|$$

$$d\lambda_1 d\lambda_2\cdots d\lambda_m$$

$$\longrightarrow 0 \qquad (10)$$

as $N \longrightarrow \infty$ for $2\omega < m-1$. However, it is equal to

$$\frac{1}{(2\pi)^{m-2}} \int_{-\pi}^{\pi}\cdots\int f(\lambda_1)f(\lambda_2)\cdots f(\lambda_{m-2})$$

$$[\frac{1}{(2\pi)^2 N} \iint_{-\pi}^{\pi} K_N^2(\lambda_{2\omega+1}+\cdots+\lambda_m)f(\lambda_{m-1})|f(\lambda_m)-f(-\lambda_{2\omega+1}-\cdots-\lambda_{m-1})| d\lambda_{m-1}d\lambda_m]$$

$$d\lambda_1 d\lambda_2\cdots d\lambda_{m-1} \qquad (11)$$

And by Schwarz Inequality,

$$\frac{1}{(2\pi)^2 N} \iint_{-\pi}^{\pi} K_N^2(\lambda_{2\omega+1}+\cdots+\lambda_m)f(\lambda_{m-1})$$

$$|f(\lambda_m)-f(-\lambda_{2\omega+1}-\cdots-\lambda_{m-1})|d\lambda_{m-1}d\lambda_m$$

$$\leq [\frac{1}{(2\pi)^2 N} \iint_{-\pi}^{\pi} K_N^2(\lambda_{2\omega+1}+\cdots+\lambda_m)f^2(\lambda_{m-1})d\lambda_{m-1}d\lambda_m]^{\frac{1}{2}}$$

$$[\frac{1}{(2\pi)^2 N} \iint_{-\pi}^{\pi} K_N^2(\lambda_{2\omega+1}+\cdots+\lambda_m)|f(\lambda_m)-f(-\lambda_{2\omega+1}-\cdots-\lambda_{m-1})|^2 d\lambda_{m-1}d\lambda_m]^{\frac{1}{2}}$$

$$\longrightarrow 0 \quad \text{uniformly for all } \lambda_{2\omega+1},\ldots,\lambda_{m-2} \text{ as } N \longrightarrow \infty$$

because the first integral is bounded by $C_2^{\frac{1}{2}}$ and the second integral goes to zero uniformly as $N \longrightarrow \infty$ by lemma (2.1). Therefore, the integral in (11) and (10) goes to zero as $N \longrightarrow \infty$, and hence (9) holds.

In the case $2\omega=m-1$, it is easy to show that

$$\frac{1}{(2\pi)^m}\frac{1}{N}\int_{-\pi}^{\pi}\cdots\int e^{i(k_{j_1}^{(\alpha)}-k_{j_2}^{(\alpha)})\lambda_1+\ldots+i(k_{j_{m-2}}^{(\alpha)}-k_{j_{m-1}}^{(\alpha)})\lambda_{\frac{m-1}{2}}}$$

$$e^{i(k_{p_1}^{(\beta)}-k_{p_2}^{(\beta)})\lambda_{\frac{m+1}{2}}+\ldots+i(k_{p_{m-2}}^{(\beta)}-k_{p_{m-1}}^{(\beta)})\lambda_{m-1}}$$

$$e^{i(k_{j_m}^{(\alpha)}-k_{p_m}^{(\beta)})\lambda_m}\quad K_N^2(\lambda_m)$$

$$f(\lambda_1)f(\lambda_2)\ldots f(\lambda_m)d\lambda_1 d\lambda_2\ldots d\lambda_m$$

$$\longrightarrow \frac{1}{(2\pi)^{m-1}}\int_{-\pi}^{\pi}\cdots\int e^{i(k_{j_1}^{(\alpha)}-k_{j_2}^{(\alpha)})\lambda_1+\ldots+i(k_{p_{m-2}}^{(\beta)}-k_{p_{m-1}}^{(\beta)})\lambda_{m-1}}$$

$$f(\lambda_1)f(\lambda_2)\ldots f(\lambda_{m-1})f(0)d\lambda_1 d\lambda_2\ldots d\lambda_{m-1}$$

as $N\longrightarrow\infty$. (12)

(9) and (12) together imply that

$$(8)\longrightarrow \sum_{A_\alpha\cup A_\beta}^{*}\frac{1}{(2\pi)^{m-1}}$$

$$\int_{-\pi}^{\pi}\cdots\int e^{i(k_{j_1}^{(\alpha)}-k_{j_2}^{(\alpha)})\lambda_1+\ldots+i(k_{j_{2\omega-1}}^{(\alpha)}-k_{j_{2\omega}}^{(\alpha)})\lambda_\omega}$$

$$e^{i(k_{p_1}^{(\beta)}-k_{p_1}^{(\beta)})\lambda_{\omega+1}+\ldots+i(k_{p_{2\omega-1}}^{(\beta)}-k_{p_{2\omega}}^{(\beta)})\lambda_{2\omega}}$$

$$e^{i(k_{j_{2\omega+1}}^{(\alpha)}-k_{p_{2\omega+1}}^{(\beta)})\lambda_{2\omega+1}+\ldots+i(k_{j_m}^{(\alpha)}-k_{p_m}^{(\beta)})(-\lambda_{2\omega+1}-\ldots-\lambda_{m-1})}$$

$$f(\lambda_1)f(\lambda_2)\ldots f(\lambda_{m-1})f(-\lambda_{2\omega+1}-\ldots-\lambda_{m-1})$$

$$d\lambda_1 d\lambda_2\ldots d\lambda_{m-1} \qquad (13)$$

where we write $f(0)$ in place of $f(-\lambda_{2\omega+1}-\lambda_{2\omega+2}-\ldots-\lambda_{m-1})$ when $2\omega=m-1$. In our notation (13) is just

$r_{\alpha,\beta}$ =

$$\frac{1}{(2\pi)^m}\int_{-\pi}^{\pi}\!\cdots\!\int \exp\begin{bmatrix}\begin{pmatrix} k_1^{(\alpha)} k_2^{(\alpha)} \ldots k_m^{(\alpha)} \\ k_1^{(\beta)} k_2^{(\beta)} \ldots k_m^{(\beta)} \end{pmatrix} \\ \lambda_1 \;\; \lambda_2 \;\; \cdots \lambda_m \end{bmatrix} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_m) d\lambda_1 d\lambda_2 \ldots d\lambda_m$$

Moreover $r_{\alpha,\beta}$ is finite because $f \in L^2(-\pi,\pi)$.

Hence, lemma (2.2) is proved.

The following 2 lemma will be needed in lemma 2.5.

Lemma 2.3 Suppose $f \in L^2(-\pi,\pi)$ and a is any real number. Then

$$\frac{1}{\sqrt{N}} \int_{-\pi}^{\pi} K_N(\lambda + a) f(\lambda) d\lambda \longrightarrow 0$$

uniformly for all a as $N \longrightarrow \infty$.

[Proof]. For each $\varepsilon > 0$, choose a $\delta(\varepsilon) > 0$ such that

$$\int_S f^2(\lambda) d\lambda < \frac{\varepsilon^2}{8\pi}$$

whenever S is a measurable set and the Lebesgue measure of S is less than $\delta(\varepsilon)$. Suppose

$$2n\pi \leq a < 2(n+1)\pi \,, \quad \text{for some integer } n$$

Let $$S_\delta = \{\lambda : \; |(\lambda + a) - 2k\pi| < \frac{\delta}{4} \,, \text{ for } k = n \text{ or } (n+1)\}$$

$$I = \{\lambda : \; -\pi \leq \lambda < \pi\}$$

$2(n-1)\pi$ $\quad$ $2n\pi$ $\quad$ a $\quad$ $2(n+1)\pi$ $\quad$ $2(n+2)\pi$

Then

$$\frac{1}{\sqrt{N}} \int_{-\pi}^{\pi} K_N(\lambda+a)f(\lambda)d\lambda$$

$$= \frac{1}{\sqrt{N}} \left(\int_{S_{\delta(\varepsilon)}} + \int_{I-S_{\delta(\varepsilon)}} \right) K_N(\lambda+a)f(\lambda)d\lambda$$

$$\leq \left[\frac{1}{N} \int_{S_{\delta(\varepsilon)}} K_N^2(\lambda+a)d\lambda \int_{S_{\delta(\varepsilon)}} f^2(\lambda)d\lambda \right]^{\frac{1}{2}} + \frac{1}{\sqrt{N}} \frac{1}{\sin\frac{1}{2}\delta(\varepsilon)} \int_{I-S_{\delta(\varepsilon)}} f(\lambda)d\lambda$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

if

$$N > \left[\frac{C_1}{\sin\frac{1}{2}\delta(\varepsilon)} \right]^2 .$$

Thus, lemma (2.3) is proved.

Lemma 2.4 Suppose $f \in L^2(-\pi,\pi)$, then

$$\frac{1}{N^{\frac{n}{2}}} \int_{-\pi}^{\pi}\!\!\cdots\!\int K_N(\pm\lambda_1\pm\lambda_2+a_1)K_N(\pm\lambda_2\pm\lambda_3+a_2)\cdots\cdots$$

$$K_N(\pm\lambda_{n-1}\pm\lambda_n+a_{n-1})K_N(\pm\lambda_n+a_n)$$

$$f(\lambda_1)f(\lambda_2)\ldots f(\lambda_n) \quad d\lambda_1 d\lambda_2\ldots d\lambda_n$$

$$\longrightarrow 0$$

uniformly for all $a_1, a_2, \ldots, a_n$ and n where n is any positive integer, and for all possible combinations of plus and minus signs appearing in the kernels, as $N \longrightarrow \infty$.

[Proof]. For each $0 < \varepsilon < 1$, by lemma (2.3) there is an $N(\varepsilon) > 0$ such that $\frac{1}{\sqrt{N}} \int_{-\pi}^{\pi} K_N(\lambda+a)f(\lambda)d\lambda < \varepsilon$ uniformly for all a whenever

$N > N(\varepsilon)$. Then

$$\frac{1}{N^{\frac{n}{2}}}\int_{-\pi}^{\pi}\cdots\int K_N(\pm\lambda_1\pm\lambda_2+a_1)K_N(\pm\lambda_2\pm\lambda_3+a_2)\ldots K_N(\pm\lambda_n+a_n)$$
$$f(\lambda_1)f(\lambda_2)\ldots f(\lambda_n)d\lambda_1 d\lambda_2\ldots d\lambda_n$$
$$< \varepsilon^n < \varepsilon$$

whenever $N > N(\varepsilon)$, by successive use of lemma (2.3). Q.E.D.

Let

$$Y_{N,\alpha} = \frac{1}{N^{\frac{1}{2}}}\sum_{t=1}^{N} X_{t_1+k_1(\alpha)}X_{t+k_2(\alpha)}\ \cdots\ X_{t+k_m(\alpha)}\ ,\quad \alpha=1,2,\ldots,n$$

Then $EY_{N,\alpha_1}Y_{N,\alpha_2}\ \cdots\ Y_{N,\alpha_\ell} = 0$, when ℓ is odd, where $\alpha_1,\alpha_2,\ldots,\alpha_\ell = 1,2,\ldots,n$, by Theorem 1. However in order to compute the moment

$$EY_{N,\alpha_1}Y_{N,\alpha_2}\cdots Y_{N,\alpha_\ell} \quad \text{when } \ell \text{ is even}$$

where $\alpha_1,\alpha_2,\ldots,\alpha_\ell = 1,2,\ldots,n$. We shall make use of the following subindex table.

$$G = \left\{\begin{matrix} t_1+k_1^{(\alpha_1)}, & t_1+k_2^{(\alpha_1)}, & \ldots, & t_1+k_m^{(\alpha_1)} \\ t_2+k_1^{(\alpha_2)}, & t_2+k_2^{(\alpha_2)}, & \ldots, & t_2+k_m^{(\alpha_2)} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ t_\ell+k_1^{(\alpha_\ell)}, & t_\ell+k_2^{(\alpha_\ell)}, & \ldots, & t_\ell+k_m^{(\alpha_\ell)} \end{matrix}\right\} \tag{14}$$

which is closely related to the limit of the above moment.

We need to introduce some new notations with respect to this set G, for ℓ even. Nevertheless, we should remark first that we shall not regard G as a set of integers depending upon the values of $t_1, t_2, \ldots, t_\ell$, but shall consider G as a set consisting of ℓm distinct elements characterized by the subindices after t and k so that each of them occupies a definite position in the table (14). For example, $t_2+k_3^{(\alpha_2)}$ is the third element in the second row of (14).

For convenience in writing, we shall put

$$
\begin{cases}
t_1+k_1^{(\alpha_1)} = s_1 \\
t_1+k_2^{(\alpha_1)} = s_2 \\
\quad \cdot\ \cdot\ \cdot\ \cdot \\
t_1+k_m^{(\alpha_1)} = s_m
\end{cases}
$$

$$
\begin{cases}
t_2+k_1^{(\alpha_2)} = s_{m+1} \\
t_2+k_2^{(\alpha_2)} = s_{m+2} \\
\quad \cdot\ \cdot\ \cdot\ \cdot \\
t_2+k_m^{(\alpha_2)} = s_{m+m} = s_{2m}
\end{cases}
\tag{15}
$$

$$\cdot\ \cdot\ \cdot\ \cdot$$

$$
\begin{cases}
t_\ell+k_1^{(\alpha_\ell)} = s_{(\ell-1)m+1} \\
t_\ell+k_2^{(\alpha_\ell)} = s_{(\ell-1)m+2} \\
\quad \cdot\ \cdot\ \cdot\ \cdot \\
t_\ell+k_m^{(\alpha_\ell)} = s_{(\ell-1)m+m} = s_{\ell m}
\end{cases}
$$

What we did above is only to change a set with two indices into a set with one index. The relation (15) gives a one-one correspondence between s's and the positions in Table (14).

Then
$$EY_{N,\alpha_1}Y_{N,\alpha_2}\cdots Y_{N,\alpha_\ell}$$

$$= E\left\{\frac{1}{N^{\frac{\ell}{2}}}\sum_{t_1=1}^{N}\cdots\sum_{t_\ell=1}^{N}(X_{t_1+k_1}^{(\alpha_1)}X_{t_1+k_2}^{(\alpha_1)}\cdots X_{t_1+k_m}^{(\alpha_1)})(X_{t_2+k_1}^{(\alpha_2)}X_{t_2+k_2}^{(\alpha_2)}\cdots X_{t_2+k_m}^{(\alpha_2)})\cdots\cdots(X_{t_\ell+k_1}^{(\alpha_\ell)}X_{t_\ell+k_2}^{(\alpha_\ell)}\cdots X_{t_\ell+k_m}^{(\alpha_\ell)})\right\}$$

$$= \frac{1}{N^{\ell/2}}\sum_{t_1=1}^{N}\cdots\sum_{t_\ell=1}^{N}E(X_{s_1}X_{s_2}\cdots X_{s_{\ell m}})$$

$$= \frac{1}{N^{\ell/2}}\sum_{t_1=1}^{N}\cdots\sum_{t_\ell=1}^{N}(\tilde{\sum_G} r_{s_{j_1}-s_{j_2}}r_{s_{j_3}-s_{j_4}}\cdots r_{s_{j_{\ell m-1}}-s_{j_{\ell m}}})$$

$$= \tilde{\sum_G}\left\{\frac{1}{N^{\ell/2}}\sum_{t_1=1}^{N}\cdots\sum_{t_\ell=1}^{N}r_{s_{j_1}-s_{j_2}}r_{s_{j_3}-s_{j_4}}\cdots r_{s_{j_{\ell m-1}}-s_{j_{\ell m}}}\right\} \tag{16}$$

There are $\frac{(\ell m)!}{2^{\ell m/2}(\frac{\ell m}{2})!}$ terms in $\tilde{\sum_G}$. Each of them is a multiple sum over $t_1,\ldots,t_\ell$ and weighted by $N^{\ell/2}$ and each sum is characterized completely by the subindices after the r's. We shall call

$$S = \left\{(s_{j_1},s_{j_2}),(s_{j_3},s_{j_4}),\ldots,(s_{j_{\ell m-1}},s_{j_{\ell m}})\right\}$$

<u>the associated subindex-pair set</u> of the multiple sum

$$\frac{1}{N^{\ell/2}} \sum_{t_1=1}^{N} \cdots \sum_{t_\ell=1}^{N} r_{s_{j_1}-s_{j_2}} r_{s_{j_3}-s_{j_4}} \cdots r_{s_{j_{\ell m-1}}-s_{j_m}}$$

Here we recall that the s_j's can be identified with the elements in Tables (14) in one-one manner by (15).

We define

$$G_1 = \{t_1+k_1^{(\alpha_1)}, t_1+k_2^{(\alpha_1)}, \ldots, t_1+k_m^{(\alpha_1)}\}$$
$$= \{s_1, s_2, \ldots, s_m\}$$
$$G_2 = \{t_2+k_1^{(\alpha_2)}, t_2+k_2^{(\alpha_2)}, \ldots, t_2+k_m^{(\alpha_2)}\}$$
$$= \{s_{m+1}, s_{m+2}, \ldots, s_{2m}\}$$
$$\cdots\cdots\cdots\cdots$$
$$G_\ell = \{t_\ell+k_1^{(\alpha_\ell)}, t_\ell+k_2^{(\alpha_\ell)}, \ldots, t_\ell+k_m^{(\alpha_\ell)}\}$$
$$= \{s_{(\ell-1)m+1}, s_{(\ell-1)m+2}, \ldots, s_{\ell m}\}$$

Hence $G = G_1 \cup G_2 \cup \ldots \cup G_\ell$ and $G_j \cap G_k = 0$, $j \neq k$. (Since we consider $t_{j_1}+k_{p_1}^{(\alpha_{j_1})}$ and $t_{j_2}+k_{p_2}^{(\alpha_{j_2})}$ to be distinct if $j_1 \neq j_2$ or $p_1 \neq p_2$). We shall also define two classes of subsets of G

$$\text{(i)} \quad H = \{H_{pq} : p < q,\ p,q=1,2,\ldots,\ell\} = \{G_p \cup G_q : p < q,\ p,q=1,2,\ldots,\ell\} \tag{17}$$

where $H_{pq} = G_p \cup G_q$

(ii) $L = \{L_j : j=1,2,\ldots,2^{\ell}-1\}$ (18)

where $L_j = G_{j_1} \cup G_{j_2} \cup \ldots \cup G_{j_n}$

for some $n < \ell$, $j_1 < j_2 \ldots < j_n$, $j_1, j_2, \ldots, j_n =$
$1,2,\ldots,\ell$.

We say an associated subindex-pair set

$$S = \{(s_{j_1}, s_{j_2}), (s_{j_3}, s_{j_4}), \ldots, (s_{j_{\ell m-1}}, s_{j_{\ell m}})\}$$

where $j_1, j_2, \ldots, j_{\ell m} = 1,2,\ldots,\ell m$ and $j_\alpha \neq j_\beta$, if $\alpha \neq \beta$, is <u>decomposable with respect to H</u>, if there are mutually disjoint subsets $S_1, S_2, \ldots, S_{\ell/2}$ of S such that

$S = \bigcup_{p=1}^{\ell/2} S_p$ and for each p, $p=1,2,\ldots,\ell/2$ we have $\cup S_p = H_{qr}$ (*)

(*)Let $A = \{a,b,c,\ldots,d\}$, by $\cup A$ we mean $\cup A = a \cup b \cup c \cup \ldots \cup d$.
Note the difference of $\bigcup_p S_p$ and $\cup S_p$

(i) $\bigcup_p S_p = S_1 \cup S_2 \cup \ldots \cup S_{\ell/2}$

(ii) $\cup S_p$ = the union of all elements of S_p

e.g. if $S_p = \{(s_1,s_2),(s_3,s_4),\ldots,(s_{2\ell-1},s_{2\ell})\}$

then $\cup S_p = \{(s_1,s_2) \cup (s_3,s_4) \cup \ldots \cup (s_{2\ell-1},s_{2\ell})\}$

$= \{ s_1,s_2,s_3,s_4,\ldots,s_{2\ell-1},s_{2\ell} \}$

for some $q,r=1,2,\ldots,\ell$. Otherwise we say S is <u>not decomposable with respect to H</u>. As an example, if

$$S = \{(s_1,s_{m+1}),(s_2,s_{m+2}),\ldots,(s_m,s_{2m})$$
$$(s_{2m+1},s_{3m+1}),(s_{2m+2},s_{3m+2}),\ldots,(s_{3m},s_{4m})$$
$$\cdots\cdots\cdots\cdots$$
$$(s_{(\ell-2)m+1},s_{(\ell-1)m+1}),\ldots,(s_{(\ell-1)m},s_{\ell m})\}$$

then $S_1 = \{(s_1,s_{m+1}),(s_2,s_{m+2}),\ldots,(s_m,s_{2m})\}$

$$\cdots\cdots\cdots\cdots$$
$$S_{\frac{\ell}{2}} = \{(s_{(\ell-2)m+1},s_{(\ell-1)m+1}),\ldots,(s_{(\ell-1)m},s_{\ell m})\}.$$

and $\cup S_1 = H_{12},\ \cup S_2 = H_{34},\ldots,\cup S_{\ell/2} = H_{\ell-1,\ell}$

hence S is decomposable with respect to H. As another example if

$$S = \{(s_{\ell m},s_{m+1}),(s_2,s_{m+2}),\ldots,(s_m,s_{2m})$$
$$(s_{2m+1},s_{3m+1}),(s_{2m+2},s_{3m+2}),\ldots,(s_{3m},s_{4m})$$
$$\cdots\cdots\cdots\cdots$$
$$(s_{(\ell-2)m+1},s_{(\ell-1)m+1}),\ldots,(s_{(\ell-1)m},s_1)\}$$

i.e. we interchange the positions of s_1 and $s_{\ell m}$ in the above example. This S is not decomposable with respect to H. Similarly, we say that S is <u>decomposable with respect to L</u> if there are mutually disjoint subsets S_1 and S_2 of S such that $S = S_1 \cup S_2$ and $\cup S_p \in L$ for p=1,2. Otherwise we say S is <u>not decomposable with respect to L</u>.

Here we see clearly that when $\ell > 2$, $H \subset L$ and hence S not decomposable with respect to L implies S is not decomposable with respect to H, but not vice versa.

Lemma 2.5 Let ℓ be a positive even integer and m be a positive odd integer; $\ell > 2$ and $m > 1$. Then, in (16),

$$\frac{1}{N^{\ell/2}} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} \cdots \sum_{t_\ell=1}^{N} r_{s_{j_1}-s_{j_2}} r_{s_{j_3}-s_{j_4}} \cdots r_{s_{j_{\ell m-1}}-s_{j_{\ell m}}} \qquad (19)$$

$$\longrightarrow 0$$

as $N \to \infty$, if its associated subindex-pair set S is not decomposable with respect to H.

[Proof]. It is sufficient to show that (19) holds if its associated subindex-pair set S is not decomposable with respect to L. Since if S is not decomposable with respect to H but is decomposable with respect to L and say, there are $S_1, S_2 \subset S$ such that

$$\cup S_1 = \{s_{j_1}, s_{j_2}, \ldots, s_{j_{pm-1}}, s_{j_{pm}}\} = G_1 \cup G_2 \cup \ldots \cup G_p$$

$$\cup S_2 = \{s_{j_{pm+1}}, s_{j_{pm+2}}, \ldots, s_{j_{\ell m+1}}, s_{j_{\ell m}}\} = G_{p+1} \cup G_{p+2} \cup \ldots \cup G_\ell$$

then our problem is reduced to proving that

$$\frac{1}{N^{p/2}} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} \cdots \sum_{t_p=1}^{N} r_{s_{j_1}-s_{j_2}} r_{s_{j_3}-s_{j_4}} \cdots r_{s_{j_{pm-1}}-s_{j_{pm}}} \qquad (20)$$

$$\longrightarrow 0$$

if $p > 3$ or

$$\frac{1}{N^{(\ell-p)/2}} \sum_{t_{p+1}=1}^{N} \sum_{t_{p+2}=1}^{N} \cdots \sum_{t_\ell=1}^{N} r_{s_{j_{pm+1}}-s_{j_{pm+2}}} r_{s_{j_{pm+3}}-s_{j_{pm+4}}} \cdots$$
$$\cdots r_{s_{j_{\ell m-1}}-s_{j_{\ell m}}} \longrightarrow 0 \qquad (21)$$

of $(\ell-p) > 3$, because in this case the summation in (19) can be factored into two parts (20) and (21) and either p or ℓ-p must be >3, and both of them ≥ 2 otherwise S would be decomposable with respect to H by the fact that p and ℓ-p must be even positive integers since pm is divisible by 2 and m is odd. Hence

$$2 \leq p,\ \ell-p \leq \ell-2 < \ell.$$

If $p > 2$, then on the part of (20), we meet the same situation as before the factorization, only now $p < \ell$. We shall factorize (20) again if its subindex-pair set is decomposable with respect to its corresponding L. We shall do the same thing to sum (21) if $\ell-p > 2$. Keeping on doing this factorization for all factors, we shall finally, through a finite number of steps, reach the following situation: Every factor has its associated subindex-pair set not decomposable with respect to its corresponding L and either (i) it is a double sum or (ii) it is a multiple sum over more than 2 indices, so it is not decomposable with respect to its corresponding H. Each factor belonging to (i), as we can show from lemma 2.2, is uniformly bounded for all N. Among all the factors, at least one of them belongs to (ii) otherwise S would be decomposable with respect to H. And if we can show all the factors belonging to (ii) tend to zero as $N \to \infty$, then our lemma is proved. Therefore, it suffices to show that (19) holds as $N \Rightarrow \infty$, if its S is not decomposable with respect to L.

Now let us consider

$$\frac{1}{N^{\ell/2}} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} \cdots \sum_{t_\ell=1}^{N} r_{s_{j_1}-s_{j_2}} r_{s_{j_3}-s_{j_4}} \cdots r_{s_{j_{\ell m-1}}-s_{j_{\ell m}}}$$

whose associated subindex-pair set S is not decomposable with respect to L. Since this multiple sum is characterized by the positions of its subindices in the subindex Table (14), we should investigate them more in detail. Let us look at the subindex table

$$\begin{array}{lllll} G_1 : & t_1+k_1^{(\alpha_1)} & t_1+k_2^{(\alpha_1)} & \cdots & t_1+k_m^{(\alpha_1)} \\ G_2 : & t_2+k_1^{(\alpha_2)} & t_2+k_2^{(\alpha_2)} & \cdots & t_2+k_m^{(\alpha_2)} \\ & \cdots\cdots\cdots\cdots & & & \\ G_\ell : & t_\ell+k_1^{(\alpha_\ell)} & t_\ell+k_2^{(\alpha_\ell)} & \cdots & t_\ell+k_m^{(\alpha_\ell)} \end{array} \tag{22}$$

We say a subindex-pair connects G_{j_1} and G_{j_2}, $j_1, j_2 = 1, 2, \ldots, \ell$ if one of its elements belongs to G_{j_1} and the other belongs to G_{j_2} e.g. $(t_1+k_1^{(\alpha_1)}, t_2+k_m^{(\alpha_2)})$ connects G_1 and G_2. We call an array of h subindex-pairs of S, $h < \ell$, a chain if there are $G_{j_1}, G_{j_2}, \ldots, G_{j_{h+1}}$, $j_a \neq j_b$ if $a \neq b$, such that the first element of the array connects G_{j_1} and G_{j_2}, the second connects G_{j_2} and G_{j_3}, ..., and the h^{th} connects G_{j_h} and $G_{j_{h+1}}$ and we say this chain connects $G_{j_1}, G_{j_2}, \ldots, G_{j_{h+1}}$. We shall call h the length of the chain. As an example,

$$(t_1+k_1^{(\alpha_1)}, t_2+k_2^{(\alpha_2)}), (t_2+k_3^{(\alpha_2)}, t_3+k_4^{(\alpha_3)}), (t_3+k_5^{(\alpha_3)}, t_4+k_5^{(\alpha_4)}),$$
$$(t_4+k_4^{(\alpha_4)}, t_5+k_3^{(\alpha_5)})$$

is a chain of length 4 which connects G_1, G_2, G_3, G_4 and G_5. Since for finite ℓ and m, the number of elements in (22) is finite, we can pick from S a longest chain which connects the greatest number of G_j's (there may be more than one such chain, we pick any one of them). Without loss of generality, we can assume that it connects $G_1, G_2, \ldots, G_{b_1}, b_1 \leq \ell$, and write it as

$$(t_1+k_{j_1}^{(\alpha_1)}, t_2+k_{j_2}^{(\alpha_2)}), (t_2+k_{j_3}^{(\alpha_2)}, t_3+k_{j_4}^{(\alpha_3)}, \ldots$$
$$\ldots, (t_{b_1-1}+k_{j_{2b_1-3}}^{(\alpha_{b_1-1})}, t_{b_1}+k_{j_{2b_1-2}}^{(\alpha_{b_1})})$$

for some $j_1, j_2, \ldots, j_{2b_1-2}=1,2,\ldots,m$, while the length of the chain is b_1-1, $b_1 \leq \ell$. Since S is not decomposable with respect to L if $b_1 < \ell$, we are to find a second chain which is a longest chain connecting one of $G_1, G_2, \ldots, G_{b_1}$ to the greatest number of other G_j's and with no loss of generality, we can assume it connects $G_{p_1}, G_{b_1+1}, G_{b_1+2}, \ldots, G_{b_2}$, with $p_1 < b_1, b_2 \leq \ell$ and write it as

$$(t_{p_1}+k_{j_{2b_1-1}}^{(\alpha_{p_1})}, t_{b_1+1}+k_{j_{2b_1}}^{(\alpha_{b_1+1})}), (t_{b_1+1}+k_{j_{2b_1+1}}^{(\alpha_{b_1+1})}, t_{b_1+2}+k_{j_{2b_1+2}}^{(\alpha_{b_1+2})}), \ldots$$
$$\ldots, (t_{b_2-1}+k_{j_{2b_2-3}}^{(\alpha_{b_2-1})}, t_{b_2}+k_{j_{2b_2-2}}^{(\alpha_{b_2})})$$

for some $j_{2b_1-1}, j_{2b_1}, \ldots, j_{2b_2-2} = 1,2,\ldots,m$, while the length of this chain is b_2-b_1. If $b_2 < \ell$, we can always find a third chain which is a longest one connecting one of $G_1, G_2, \ldots, G_{b_2}$ to the greatest number of other G_j's. Keeping on doing this, because S is not decomposable with respect to L and ℓ is finite, we shall find a $(\omega+1)^{th}$ chain such that it is one of the chains connecting one of $G_1, G_2, \ldots, G_{b_\omega}$ to $G_{b_\omega+1}, G_{b_\omega+2}, \ldots, G_\ell$, we write it as

$$(t_{p_\omega} + k_{j_{2b_\omega-1}}^{(\alpha_{p_\omega})}, t_{b_\omega+1} + k_{j_{2b_\omega}}^{(\alpha_{b_\omega+1})}), (t_{b_\omega+1} + k_{j_{2b_\omega+1}}^{(\alpha_{b_\omega+1})}, t_{b_\omega+2} + k_{j_{2b_\omega+2}}^{(\alpha_{b_\omega+2})}), \ldots$$

$$\ldots, (t_{\ell-1} + k_{j_{2\ell-3}}^{(\alpha_{\ell-1})}, t_\ell + k_{j_{2\ell-2}}^{(\alpha_\ell)})$$

where $p_\omega < b$ and $j_{2b_\omega-1}, j_{2b_\omega}, \ldots, j_{2\ell-2}$ are some numbers from $1,2,\ldots,m$ and the length of this chain is $\ell - b_\omega$. Apparently the number of chains, picked in this way, is $\omega+1$ where $\omega+1 < \ell$ and the total number of subindex-pairs in these chains is $\ell-1$. Note that the lengths of the chains are non-increasing and some of them may contain only one pair. However, the length of the first chain is always ≥ 2.

Table of Chains

first chain : $(t_1+k_{j_1}^{(\alpha_1)}, t_2+k_{j_2}^{(\alpha_2)}), (t_2+k_{j_3}^{(\alpha_2)}, t_3+k_{j_4}^{(\alpha_3)}), \ldots$

$\ldots, (t_{b_1-1}+k_{j_{2b_1-3}}^{(\alpha_{b_1-1})}, t_{b_1}+k_{j_{2b_1-2}}^{(\alpha_{b_1})})$

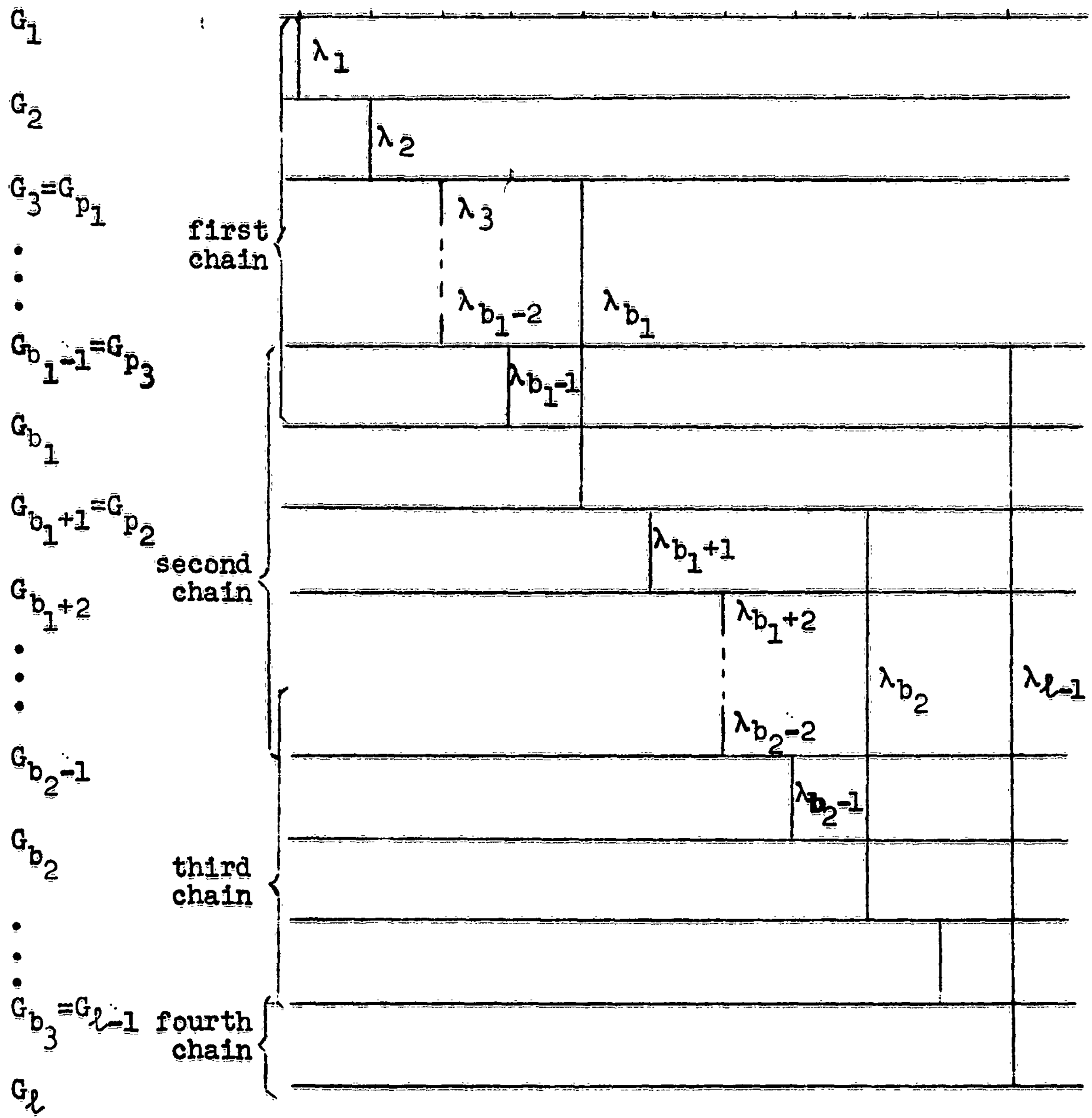
second chain: $(t_{p_1}+k_{j_{2b_1-1}}^{(\alpha_{p_1})}, t_{b_1+1}+k_{j_{2b_1}}^{(\alpha_{b_1+1})}), (t_{b_1+1}+k_{j_{2b_1+1}}^{(\alpha_{b_1+1})},$

$t_{b_1+2}+k_{j_{2b_1+2}}^{(\alpha_{b_1+2})}), \ldots, (t_{b_2-1}+k_{j_{2b_2-3}}^{(\alpha_{b_2-1})}, t_{b_2}+k_{j_{2b_2-2}}^{(\alpha_{b_2})})$

$(\omega+1)^{th}$ chain: $(t_{p_\omega}+k_{j_{2b_\omega-1}}^{(\alpha_{p_\omega})}, t_{b_\omega+1}+k_{j_{2b_\omega}}^{(\alpha_{b_\omega+1})}), (t_{b_\omega+1}+k_{j_{2b_\omega+1}}^{(\alpha_{b_\omega+1})},$

$t_{b_\omega+2}+k_{j_{2b_\omega+2}}^{(\alpha_{b_\omega+2})}), \ldots, (t_{\ell-1}+k_{j_{2\ell-3}}^{(\alpha_{\ell-1})}, t_\ell+k_{j_{2\ell-2}}^{(\alpha_\ell)})$

An illustrating diagram of chains (for $\omega=3$, $\omega+1=4$)



And we shall write the other $\frac{\ell(m-2)+2}{2}$ pairs as

$$(t_{q_{2\ell-1}}+k_{j_{2\ell-1}}^{(\alpha_{q_{2\ell-1}})}, t_{q_{2\ell}}+k_{j_{2\ell}}^{(\alpha_{q_{2\ell}})}), (t_{q_{2\ell+1}}+k_{j_{2\ell+1}}^{(\alpha_{q_{2\ell+1}})}, t_{q_{2\ell+2}}+k_{j_{2\ell+2}}^{(\alpha_{q_{2\ell+2}})})$$

$$\ldots, (t_{q_{\ell m-1}}+q_{j_{\ell m-1}}^{(\alpha_{q_{\ell m-1}})}, t_{q_{\ell m}}+k_{j_{\ell m}}^{(\alpha_{q_m})})$$

where $q_n = 1,2,\ldots,\ell$ for $n = 2\ell-1, 2\ell, \ldots, \ell m$

$j_n = 1,2,\ldots,m$ for $n = 2\ell-1, 2\ell, \ldots, \ell m$.

Then

$$\frac{1}{N^{\ell/2}} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} \ldots \sum_{t_\ell=1}^{N} r_{s_{j_1}-s_{j_2}} r_{s_{j_3}-s_{j_4}} \ldots r_{s_{j_{\ell m-1}}-s_{j_{\ell m}}}$$

$$= \frac{1}{N^{\ell/2}} \sum_{t_1=1}^{N} \ldots \sum_{t_\ell=1}^{N} r_{t_1+k_{j_1}^{(\alpha_1)}-t_2-k_{j_2}^{(\alpha_2)}} r_{t_2+k_{j_3}^{(\alpha_2)}-t_3-k_{j_4}^{(\alpha_3)}} \ldots$$

$$\ldots r_{t_{b_1-1}+k_{j_{2b_1-3}}^{(\alpha_{b_1-1})}-t_{b_1}-k_{j_{2b_1-2}}^{(\alpha_{b_1})}}$$

$$r_{t_{p_1}+k_{j_{2b_1-1}}^{(\alpha_{p_1})}-t_{b_1+1}-k_{j_{2b_1}}^{(\alpha_{b_1+1})}} r_{t_{b_1+1}+k_{j_{2b_1+1}}^{(\alpha_{b_1+1})}-t_{b_1+2}-k_{j_{2b_1+2}}^{(\alpha_{b_1+2})}} \ldots$$

$$\ldots r_{t_{b_2-1}+k_{j_{2b_2-3}}^{(\alpha_{b_2-1})}-t_{b_2}-k_{j_{2b_2-2}}^{(\alpha_{b_2})}}$$

. . . .

$$r_{t_{p_\omega}+k_{j_{2b_\omega-1}}^{(\alpha_{p_\omega})}-t_{b_\omega+1}-k_{j_{2b_\omega}}^{(\alpha_{b_\omega+1})}}\, r_{t_{b_\omega+1}+k_{j_{2b_\omega+1}}^{(\alpha_{b_\omega+1})}-t_{b_\omega+2}+k_{j_{2b_\omega+2}}^{(\alpha_{b_\omega+2})}} \cdots$$

$$\cdots r_{t_{\ell-1}+k_{j_{2\ell-3}}^{(\alpha_{\ell-1})}-t_\ell-k_{j_{2\ell-2}}^{(\alpha_\ell)}}$$

$$r_{t_{q_{2\ell-1}}+k_{j_{2\ell-1}}^{(\alpha_{q_{2\ell-1}})}-t_{q_{2\ell}}-k_{j_{2\ell}}^{(\alpha_{q_{2\ell}})}}\, r_{t_{q_{2\ell+1}}+k_{j_{2\ell+1}}^{(\alpha_{q_{2\ell+1}})}-t_{q_{2\ell+2}}-k_{j_{2\ell+2}}^{(\alpha_{q_{2\ell+2}})}} \cdots$$

$$\cdots r_{t_{q_{\ell m-1}}+k_{j_{\ell m-1}}^{(\alpha_{q_{\ell m-1}})}-t_{q_{\ell m}}-k_{j_{\ell m}}^{(\alpha_{q_{\ell m}})}}$$

$$= \frac{1}{N^{\ell/2}} \sum_{t_1=1}^{N} \cdots \sum_{t_\ell=1}^{N} \frac{1}{(2\pi)^{\ell m/2}} \int_{-\pi}^{\pi} e^{i(t_1+k_{j_1}^{(\alpha_1)}-t_2-k_{j_2}^{(\alpha_2)})\lambda_1} f(\lambda_1)\,d\lambda_1$$

$$\int_{-\pi}^{\pi} e^{i(t_2+k_{j_3}^{(\alpha_2)}-t_3-k_{j_4}^{(\alpha_3)})\lambda_2} f(\lambda_2)\,d\lambda_2 \cdots$$

$$\cdots \int_{-\pi}^{\pi} e^{i(t_{b_1-1}+k_{j_{2b_1-3}}^{(\alpha_{b_1-1})}-t_{b_1}-k_{j_{2b_1-2}}^{(\alpha_{b_1})})\lambda_{b_1-1}} f(\lambda_{b_1-1})\,d\lambda_{b_1-1}$$

$$\int_{-\pi}^{\pi} e^{i(t_{p_1}+k_{j_{2b_1-1}}^{(\alpha_{p_1})}-t_{b_1+1}-k_{j_{2b_1}}^{(\alpha_{b_1+1})})\lambda_{b_1}} f(\lambda_{b_1})\,d\lambda_{b_1}$$

$$\int_{-\pi}^{\pi} e^{i(t_{b_1+1}+k_{j_{2b_1+1}}^{(\alpha_{b_1+1})}-t_{b_1+2}-k_{j_{2b_1+2}}^{(\alpha_{b_1+2})})\lambda_{b_1+1}} f(\lambda_{b_1+1})\,d\lambda_{b_1+1} \cdots$$

$$\cdots \int_{-\pi}^{\pi} e^{i(t_{b_2-1}+k_{j_{2b_2-3}}^{(\alpha_{b_2-1})}-t_{b_2}-k_{j_{2b_2-2}}^{(\alpha_{b_2})})\lambda_{b_2-1}} f(\lambda_{b_2-1})d\lambda_{b_2-1}$$

$$\cdots \int_{-\pi}^{\pi} e^{i(t_{\ell-1}+k_{j_{2\ell-3}}^{(\alpha_{\ell-1})}-t_{\ell}-k_{2\ell-2}^{(\alpha_{\ell})})\lambda_{\ell-1}} f(\lambda_{\ell-1})d\lambda_{\ell-1}$$

$$\int_{-\pi}^{\pi} e^{i(t_{q_{2\ell-1}}+k_{j_{2\ell-1}}^{(\alpha_{q_{2\ell-1}})}-t_{q_{2\ell}}-k_{j_{2\ell}}^{(\alpha_{q_{2\ell}})})\lambda_{\ell}} f(\lambda_{\ell})d\lambda_{\ell}$$

$$\int_{-\pi}^{\pi} e^{i(t_{q_{2\ell+1}}+k_{j_{2\ell+1}}^{(\alpha_{q_{2\ell+1}})}-t_{q_{2\ell+2}}-k_{j_{2\ell+2}}^{(\alpha_{q_{2\ell+2}})})\lambda_{\ell+1}} f(\lambda_{\ell+1})d\lambda_{\ell+1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\int_{-\pi}^{\pi} e^{i(t_{q_{\ell m-1}}+k_{j_{\ell m-1}}^{(\alpha_{q_{\ell m-1}})}-t_{q_{\ell m}}-k_{j_{\ell m}}^{(\alpha_{q_{\ell m}})})\lambda_{\ell m/2}} f(\lambda_{\ell m/2})d\lambda_{\ell m/2}$$

We like to rearrange the order of integration and regroup the exponentials according to the t_j's. In this arrangement we can only write out some of the λ_j's explicitly which are essential to the proof and we shall denote by some $u_{j,k}$'s the rest of the λ_j's whose exact positions in the following expression are irrelevant to our argument. We have, then,

$$\frac{1}{N^{\ell/2}} \frac{1}{(2\pi)^{\ell m/2}} \sum_{t_1=1}^{N} \cdots \sum_{t_\ell=1}^{N}$$

$$\int_{-\pi}^{\pi} \cdots \int e^{it_1(\lambda_1+u_{1,1}+u_{1,2}+\cdots+u_{1,m-1})}$$

$$e^{it_2(-\lambda_1+\lambda_2+u_{2,1}+u_{2,2}+\cdots+u_{2,m-2})}$$

$$\cdots\cdots\cdots$$

$$e^{it_{b_1-1}(-\lambda_{b_1-2}+\lambda_{b_1-1}+u_{b_1-1,1}+u_{b_1-1,2}+\cdots+u_{b_1-1,m-2})}$$

$$e^{it_{b_1}(-\lambda_{b_1-1}+u_{b_1,1}+u_{b_1,2}+\cdots+u_{b_1,m-1})}$$

$$e^{it_{b_1+1}(-\lambda_{b_1}+\lambda_{b_1+1}+u_{b_1+1,1}+u_{b_1+1,2}+\cdots+u_{b_1+1,m-2})}$$

$$\cdots\cdots\cdots$$

$$e^{it_\ell(-\lambda_{\ell-1}+u_{\ell,1}+u_{\ell,2}+\cdots+u_{\ell,m-1})}$$

$$e^{i(k_{j_1}^{(\alpha_1)}-k_{j_2}^{(\alpha_2)})\lambda_1+i(k_{j_3}^{(\alpha_2)}-k_{j_4}^{(\alpha_3)})\lambda_2+\cdots \cdots+i(k_{j_{\ell m-1}}^{(\alpha_{q_{\ell m-1}})}-k_{j_{\ell m}}^{(\alpha_{q_{\ell m}})})\lambda_{\ell m/2}}$$

$$f(\lambda_1)f(\lambda_2)\cdots f(\lambda_{\ell m/2})d\lambda_1 d\lambda_2\cdots d\lambda_{\ell m/2}$$

where

$$u_{1,1}, u_{1,2}, \ldots, u_{1,m-1}$$

$$u_{2,1}, u_{2,2}, \ldots, u_{2,m-2}$$

$$\cdots\cdots\cdots$$

$$u_{\ell,1}, u_{\ell,2}, \ldots, u_{\ell,m-1} = \pm\lambda_\ell, \pm\lambda_{\ell+1}, \ldots, \pm\lambda_{\ell m/2}$$

By changing the order of summation and integration, we have

$$\frac{1}{N^{\ell/2}} \frac{1}{(2\pi)^{\ell m/2}} \int_{-\pi}^{\pi}\!\!\cdots\!\!\int \frac{\sin \frac{N}{2}(\lambda_1+u_{1,1}+u_{1,2}+\dots+u_{1,m-1})}{\sin \frac{1}{2}(\lambda_1+u_{1,1}+u_{1,2}+\dots+u_{1,m-1})}$$

$$\frac{\sin \frac{N}{2}(-\lambda_1+\lambda_2+u_{2,1}+u_{2,2}+\dots+u_{2,m-2})}{\sin \frac{1}{2}(-\lambda_1+\lambda_2+u_{2,1}+u_{2,2}+\dots+u_{2,m-2})}$$

$$\cdots\cdots\cdots$$

$$\frac{\sin \frac{N}{2}(-\lambda_{b_1-1}+u_{b_1,1}+u_{b_1,2}+\dots+u_{b_1,m-1})}{\sin \frac{1}{2}(-\lambda_{b_1-1}+u_{b_1,1}+u_{b_1,2}+\dots+u_{b_1,m-1})}$$

$$\frac{\sin \frac{N}{2}(-\lambda_{b_1}+\lambda_{b_1+1}+u_{b_1+1,1}+u_{b_1+1,2}+\dots+u_{b_1+1,m-2})}{\sin \frac{1}{2}(-\lambda_{b_1}+\lambda_{b_1+1}+u_{b_1+1,1}+u_{b_1+1,2}+\dots+u_{b_1+1,m-2})}$$

$$\cdots\cdots\cdots$$

$$\frac{\sin \frac{N}{2}(-\lambda_{\ell-1}+u_{\ell,1}+u_{\ell,2}+\dots+u_{\ell,m-1})}{\sin \frac{1}{2}(-\lambda_{\ell-1}+u_{\ell,1}-u_{\ell,2}+\dots+u_{\ell,m-1})}$$

$$e^{i(k_{j_1}^{(\alpha_1)}-k_{j_2}^{(\alpha_2)})\lambda_1+\dots+i(k_{j_{\ell m-1}}^{(\alpha_{q_{\ell m-1}})}-k_{j_{\ell m}}^{(\alpha_{q_{\ell m}})})\lambda_{\ell m/2}}$$

$$f(\lambda_1)f(\lambda_2)\dots f(\lambda_{\ell m/2})d\lambda_1 d\lambda_2\dots d\lambda_{\ell m/2} \qquad (23)$$

Remark. (i) In $(\lambda_1+u_{1,1}+\dots+u_{1,m-1}),(-\lambda_1+\lambda_2+u_{2,1}+\dots+u_{2,m-2}),\dots,$ $(-\lambda_{\ell-1}+u_{\ell,1}+\dots+u_{\ell,m-1})$ there are exactly one λ_j and one $-\lambda_j$ for each j hence, $(\lambda_1+u_{1,1}+\dots+u_{1,m-1})+\dots+(-\lambda_{\ell-1}+u_{\ell,1}+\dots+u_{\ell,m-1})$ $= 0$.

(ii) No two of $|\lambda_1+u_{1,1}+\ldots+u_{1,m-1}|$, $|-\lambda_1+\lambda_2+u_{2,1}+\ldots+u_{2,m-2}|$, $\ldots$, $|-\lambda_{\ell-1}+u_{\ell,1}+\ldots+u_{\ell,m-1}|$ are identically equal, because if any two of them were identically equal, it would imply S is decomposable with respect to L which would contradict our assumption.

(iii) Because m is odd,

$$\lambda_1+u_{1,1}+\ldots+u_{1,m-1} \not\equiv 0$$
$$-\lambda_1+\lambda_2+u_{2,1}+\ldots+u_{2,m-2} \not\equiv 0$$
$$-\lambda_{\ell-1}+u_{\ell,1}+\ldots+u_{\ell,m-1} \not\equiv 0$$

Our goal is to show that the integral in (23) $\longrightarrow 0$ as $N \longrightarrow \infty$. The absolute value of the integral in (23) is less than

$$\begin{aligned} \frac{1}{N^{\ell/2}} \frac{1}{(2\pi)^{\ell m/2}} \int_{-\pi}^{\pi}\!\!\cdots\!\int \; & K_N(\lambda_1+u_{1,1}+\ldots+u_{1,m-1}) \\ & K_N(-\lambda_1+\lambda_2+u_{2,1}+\ldots+u_{2,m-2}) \\ & \cdots\cdots\cdots\cdots \\ & K_N(-\lambda_{b_1-1}+u_{b_1,1}+\ldots+u_{b_1,m-1}) \\ & K_N(-\lambda_{b_1}+\lambda_{b_1+1}+u_{b_1+1,1}+\ldots+u_{b_1+1,m-2}) \\ & K_N(-\lambda_{b_1+1}+\lambda_{b_1+2}+u_{b_1+2,1}+\ldots+u_{b_1+2,m-2}) \\ & \cdots\cdots\cdots\cdots \\ & K_N(-\lambda_{\ell-1}+u_{\ell,1}+\ldots+u_{\ell,m-1}) \\ & f(\lambda_1)f(\lambda_2)\ldots f(\lambda_{\ell m/2})d\lambda_1 d\lambda_2\ldots d\lambda_{\ell m/2} \end{aligned} \tag{24}$$

Since the integral in (24) is finite for each N, by Fubini's theorem we can rearrange the order of integration. Since

$$K_N(\lambda_1+u_{1,1}+\dots+u_{1,m-1})K_N(-\lambda_{\ell-1}+u_{\ell,1}+\dots+u_{\ell,m-1})$$
$$\leq \frac{1}{2}[K_N^2(\lambda_1+u_{1,1}+\dots+u_{1,m-1})+K_N^2(-\lambda_{\ell-1}+u_{\ell,1}+\dots+u_{\ell,m-1})],$$

(24) is less than

$$\frac{1}{2}\Bigg\{\frac{1}{N^{\ell/2}}\frac{1}{(2\pi)^{\ell m/2}}\int_{-\pi}^{\pi}\!\!\cdots\!\!\int K_N^2(\lambda_1+u_{1,1}+\dots+u_{1,m-1})$$
$$K_N(-\lambda_1+\lambda_2+u_{2,1}+\dots+u_{2,m-2})$$
$$\cdots\cdots\cdots$$
$$K_N(-\lambda_{\ell-2}+\lambda_{\ell-1}+u_{\ell-1,1}+\dots+u_{\ell-1,m-2})$$
$$f(\lambda_1)f(\lambda_2)\dots f(\lambda_{\ell m/2})\,d\lambda_1 d\lambda_2\dots d\lambda_{\ell m/2}$$
$$+\frac{1}{N^{\ell/2}}\frac{1}{(2\pi)^{\ell m/2}}\int_{-\pi}^{\pi}\!\!\cdots\!\!\int K_N(-\lambda_1+\lambda_2+u_{2,1}+\dots+u_{2,m-2})$$
$$\cdots\cdots\cdots$$
$$K_N(-\lambda_{\ell-2}+\lambda_{\ell-1}+u_{\ell-1,1}+\dots+u_{\ell-1,m-2})$$
$$K_N^2(-\lambda_{\ell-1}+u_{\ell,1}+\dots+u_{\ell,m-1})$$
$$f(\lambda_1)f(\lambda_2)\dots f(\lambda_{\ell m/2})d\lambda_1 d\lambda_2\dots d\lambda_{\ell m/2}\Bigg\} \tag{25}$$

If the last chain is of length 1, $K_N(-\lambda_{\ell-2}+\lambda_{\ell-1}+u_{\ell-1,1}+\dots$ $\dots+u_{\ell-1,m-2})$ in (25) should be replaced by $K_N(-\lambda_{\ell-2}+u_{\ell-1,1}+\dots$ $\dots+u_{\ell-1,m-1})$. In the second part of (25)

(i) $$\frac{1}{(2\pi)^{(\ell-2)}N^{(\ell-2)/2}} \int_{-\pi}^{\pi}\!\!\cdots\!\!\int K_N(-\lambda_1+\lambda_2+u_{2,1}+\cdots+u_{2,m-2})$$
$$K_N(-\lambda_2+\lambda_3+u_{3,1}+\cdots+u_{3,m-2})$$
$$\cdots\cdots\cdots\cdots$$
$$K_N(-\lambda_{\ell-2}+\lambda_{\ell-1}+u_{\ell-1,1}+\cdots+u_{\ell-1,m-2})$$
$$f(\lambda_1)f(\lambda_2)\cdots f(\lambda_{\ell-2})d\lambda_1 d\lambda_2\cdots d\lambda_{\ell-2} \tag{26}$$

$\longrightarrow 0$

uniformly for all $\lambda_{\ell-1},\lambda_\ell,\lambda_{\ell+1},\ldots,\lambda_{\ell m/2}$ as $N\to\infty$ by lemma (2.4).

(ii) the remaining part

$$\frac{1}{(2\pi)^{\frac{\ell m}{2}-\ell+2}}\frac{1}{N}\int_{-\pi}^{\pi}\!\!\cdots\!\!\int K_N^2(-\lambda_{\ell-1}+u_{\ell,1}+\cdots+u_{\ell,m-1})$$
$$f(\lambda_{\ell-1})f(\lambda_\ell)f(\lambda_{\ell+1})\cdots f(\lambda_{\ell m/2})d\lambda_{\ell-1}d\lambda_\ell\cdots d\lambda_{\ell m/2}$$
$$< \max[C_3C_1^{\frac{\ell m}{2}-\ell},\ C_2C_1^{\frac{\ell m}{2}-\ell-1}]$$

hence the second part of (25) approaches zero as $N\to\infty$. We can show the first part of (25) also approaches to zero as $N\to\infty$ by a similar argument.

Therefore (25) $\to 0$ as $N\to\infty$. It follows that (24), and hence (23), approaches zero as $N\to\infty$. Thus lemma (2.5) is complete.

<u>Lemma 2.6</u> Let m be a positive odd integer and $\alpha_1,\alpha_2,\ldots,\alpha_\ell=1,2,\ldots,n$. Let $A=\{\alpha_1,\alpha_2,\ldots,\alpha_\ell\}$. Then

$$E(Y_{N,\alpha_1} Y_{N,\alpha_2} \cdots Y_{N,\alpha_\ell})$$

$$\begin{cases} \equiv 0 & \text{for all } N \quad \text{when } \ell \text{ is odd} \quad \text{(a)} \\ \longrightarrow \tilde{\sum_A} r_{\alpha_{j_1},\alpha_{j_2}} r_{\alpha_{j_3},\alpha_{j_4}} \cdots r_{\alpha_{j_{\ell-1}},\alpha_{j_\ell}} & \text{when } \ell \text{ is even} \quad \text{(b)} \end{cases}$$

as $N \to \infty$, where $\tilde{\sum_A}$ was defined in Theorem 1.

[Proof].

The proof of part (a) is very simple.

$$E\left\{\frac{1}{N^{\ell/2}} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} \cdots \sum_{t_\ell=1}^{N} (X_{t_1+k_1}(\alpha_1) X_{t_1+k_2}(\alpha_1) \cdots X_{t_1+k_m}(\alpha_1)) \right.$$

$$(X_{t_2+k_1}(\alpha_2) X_{t_2+k_2}(\alpha_2) \cdots X_{t_2+k_m}(\alpha_2))$$

$$\cdots\cdots\cdots\cdots$$

$$\left. (X_{t_\ell+k_1}(\alpha_\ell) X_{t_\ell+k_2}(\alpha_\ell) \cdots X_{t_\ell+k_m}(\alpha_\ell)) \right\}$$

$$= \frac{1}{N^{\ell/2}} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} \cdots \sum_{t_\ell=1}^{N} E\left\{(X_{t_1+k_1}(\alpha_1) X_{t_1+k_2}(\alpha_1) \cdots X_{t_1+k_m}(\alpha_1)) \right.$$

$$\cdots\cdots\cdots\cdots$$

$$\left. (X_{t_\ell+k_1}(\alpha_\ell) X_{t_\ell+k_2}(\alpha_\ell) \cdots X_{t_\ell+k_m}(\alpha_\ell)) \right\}$$

$$\equiv 0 \qquad \text{for all } N,$$

by Theorem 1, because ℓ and m are odd implies ℓm is odd.

The proof of part (b) for the case m=1 is clear. The part (b) of the lemma for m > 1 also follows easily from lemma (2.2) and lemma (2.5). Lemma (2.5) tells us in (16) only those terms whose subindex-pair set are decomposable with respect to H have non-zero limits and lemma (2.2) gives us the limits of these terms which is just our desired result.

Hence lemma (2.6) is complete.

Lemma 2.7

(a) $$E\left(\sum_{\alpha=1}^{n} \mu_\alpha Y_{N,\alpha}\right)^{2\ell} \longrightarrow \frac{(2\ell)!}{2^\ell \ell!}\left[\sum_{\alpha_1=1}^{n}\sum_{\alpha_2=1}^{n} \mu_{\alpha_1}\mu_{\alpha_2} r_{\alpha_1,\alpha_2}\right]^\ell$$

$$= \frac{(2\ell)!}{2^\ell \ell!}\left[\lim_{N\to} E\left(\sum_{\alpha=1}^{n}\mu_\alpha Y_{N,\alpha}\right)^2\right]^\ell$$

(b) $$E\left(\sum_{\alpha=1}^{n} \mu_\alpha Y_{N,\alpha}\right)^{2\ell+1} \rightarrow 0$$

as $N \to \infty$, where $\ell=0,1,2,\ldots$ and μ_α, $\alpha=1,2,\ldots,n$ are any real numbers.

[Proof]. (a) Let $A = \{\alpha_1,\alpha_2,\ldots,\alpha_{2\ell}\}$

$$E\left(\sum_{\alpha=1}^{n} \mu_\alpha Y_{N,\alpha}\right)^{2\ell}$$

$$= \sum_{\alpha_1=1}^{n}\sum_{\alpha_2=1}^{n}\cdots\sum_{\alpha_{2\ell}=1}^{n} \mu_{\alpha_1}\mu_{\alpha_2}\cdots\mu_{\alpha_{2\ell}} E(Y_{N,\alpha_1}Y_{N,\alpha_2}\cdots Y_{N,\alpha_{2\ell}})$$

$$\Rightarrow \sum_{\alpha_1=1}^{n}\sum_{\alpha_2=1}^{n}\cdots\sum_{\alpha_{2\ell}=1}^{n} \mu_{\alpha_1}\mu_{\alpha_2}\cdots\mu_{\alpha_{2\ell}} \tilde{\sum_{A}} r_{\alpha_{j_1},\alpha_{j_2}} r_{\alpha_{j_3},\alpha_{j_4}}\cdots r_{\alpha_{j_{2\ell-1}},\alpha_{j_{2\ell}}} \tag{27}$$

by lemma 2.6. Note that $(\alpha_{j_1},\alpha_{j_2},\ldots,\alpha_{j_{2\ell}})$ are all permutations of $(\alpha_1,\alpha_2,\ldots,\alpha_{2\ell})$. (27) is equal to

$$\tilde{\sum_A}\left[\sum_{\alpha_1=1}^{n}\sum_{\alpha_2=1}^{n}\cdots\sum_{\alpha_{2\ell}=1}^{n}\mu_{\alpha_1}\mu_{\alpha_2}\cdots\mu_{\alpha_{2\ell}}\, r_{\alpha_{j_1},\alpha_{j_2}}\, r_{\alpha_{j_3},\alpha_{j_4}}\cdots r_{\alpha_{j_{2\ell-1}},\alpha_{j_{2\ell}}}\right]$$

$$= \tilde{\sum_A}\left[\left(\sum_{\alpha_{j_1}=1}^{n}\sum_{\alpha_{j_2}=1}^{n}\mu_{\alpha_{j_1}}\mu_{\alpha_{j_2}}r_{\alpha_{j_1},\alpha_{j_2}}\right)\right.$$

$$\left(\sum_{\alpha_{j_3}=1}^{n}\sum_{\alpha_{j_4}=1}^{n}\mu_{\alpha_{j_3}}\mu_{\alpha_{j_4}}r_{\alpha_{j_3},\alpha_{j_4}}\right)\cdots$$

$$\left.\cdots\left(\sum_{\alpha_{j_{2\ell}}=1}^{n}\sum_{\alpha_{j_{2\ell}}=1}^{n}\mu_{\alpha_{j_{2\ell-1}}}\mu_{\alpha_{j_{2\ell}}}r_{\alpha_{j_{2\ell-1}},\alpha_{j_{2\ell}}}\right)\right]$$

But each factor in the summand

$$\sum_{\alpha_{j_{2p-1}}=1}^{n}\sum_{\alpha_{j_{2p}}=1}^{n}\mu_{\alpha_{j_{2p-1}}}\mu_{\alpha_{j_{2p}}}r_{\alpha_{j_{2p-1}},\alpha_{j_{2p}}}$$

$$= \sum_{\alpha_1=1}^{n}\sum_{\alpha_2=1}^{n}\mu_{\alpha_1}\mu_{\alpha_2}r_{\alpha_1,\alpha_2}$$

and there are $\frac{(2\ell)!}{2^{\ell}\ell!}$ terms in $\tilde{\sum_A}$. Hence (27) equals to

$$\frac{(2\ell)!}{2^{\ell}\ell!}\left[\sum_{\alpha_1=1}^{n}\sum_{\alpha_2=1}^{n}\mu_{\alpha_1}\mu_{\alpha_2}r_{\alpha_1,\alpha_2}\right]^{\ell}$$

which is our desired result.

(b) The assertion of (b) follows directly from lemma 2.6.

Hence lemma 2.7 is proved.

[Proof of Theorem 2]

Let

$$Y_{N,\alpha} = \frac{1}{N^{\frac{1}{2}}} \sum_{t=1}^{N} X_{t+k_1(\alpha)} X_{t+k_2(\alpha)} \cdots X_{t+k_m(\alpha)}$$

and let

$$Y_N = \sum_{\alpha=1}^{n} \mu_\alpha Y_{N,\alpha}$$

be any linear combination of $Y_{N,\alpha}$, $\alpha=1,2,\ldots,n$, where μ_α, $\alpha=1,2,\ldots,n$ are any real numbers.

We have shown in lemma 2.7 that all the moments of Y_N converges to the moments of a normal variable with mean zero and variance

$$\sum_{\alpha_1=1}^{n} \sum_{\alpha_2=1}^{n} \mu_{\alpha_1}\mu_{\alpha_2} r_{\alpha_1,\alpha_2} \qquad (28)$$

By moments convergence theorem [7], the distribution function of Y_N converges weakly to that of a normal variable with mean zero and variance (28). Hence $Y_{N,\alpha}$, $\alpha=1,2,\ldots,n$ is asymptotically, jointly normally distributed with mean zero and the desired covariances.

Q.E.D.

Remark: From the proof, we see that the places where we need m to be odd and $Y_{N,\alpha}$, $\alpha=1,2,\ldots,n$ to be of the same degree m are

$$(1) \quad EX_{t+k_1(\alpha)} \cdots X_{t+k_m(\alpha)} = 0$$

and (ii) $$\begin{cases} \lambda_1+u_{1,1}+\dots+u_{1,m-1} \not\equiv 0 \\ \dots\dots\dots \\ -\lambda_{\ell-1}+u_{\ell,1}+\dots+u_{\ell,m-1} \not\equiv 0 \end{cases}$$

in lemma 2.5.

(i) can be justified easily when m is not odd by subtracting the means and we shall see that after subtracting the means (ii) is also justified, therefore Theorem 2 can easily be generalized to the case that $Y_{N,\alpha}$, $\alpha=1,2,\dots,n$ are of degrees m_α, respectively where m_α, $\alpha=1,2,\dots,n$ are any positive integers.

3. The Main Theorem.

Theorem 3 Suppose $f \in L^2(-\pi,\pi)$ and

$$\lim_{N\to\infty} \frac{1}{2\pi N} \int_{-\pi}^{\pi} K_N^2(\lambda) f(\lambda)\, d\lambda \tag{29}$$

exists and is finite. Then

$$Y_{N,\alpha} = \frac{1}{N^{\frac{1}{2}}} \sum_{t_1=1}^{N} (X_{t+k_1^{(\alpha)}} X_{t+k_2^{(\alpha)}} \cdots X_{t+k_{m_\alpha}^{(\alpha)}} - EX_{t+k_1^{(\alpha)}} \cdots X_{t+k_{m_\alpha}^{(\alpha)}})$$

$\alpha=1,2,\ldots,n$, where n is a positive integer and m_α are positive integers and $k_1^{(\alpha)},\ldots,k_{m_\alpha}^{(\alpha)}$ are integers, are asymptotically jointly normally distributed with mean zero and covariances

$$r_{\alpha,\beta} = \begin{cases} 0 & \text{if } m_\alpha+m_\beta \text{ is odd} \\ \dfrac{1}{(2\pi)^{\frac{m_\alpha+m_\beta}{2}}} \displaystyle\int_{-\pi}^{\pi}\!\!\cdots\!\int \exp \begin{bmatrix} \begin{pmatrix} k_1^{(\alpha)}, \ldots, k_{m_\alpha}^{(\alpha)} \\ k_1^{(\beta)}, \ldots, k_{m_\beta}^{(\beta)} \end{pmatrix} \\ \lambda_1 \quad ,\ldots, \lambda_{\frac{m_\alpha+m_\beta}{2}} \end{bmatrix} f(\lambda_1)\cdots f(\lambda_{\frac{m_\alpha+m_\beta}{2}}) \\ \qquad d\lambda_1 d\lambda_2 \cdots d\lambda_{\frac{m_\alpha+m_\beta}{2}} & \text{if } m_\alpha+m_\beta \text{ is even} \end{cases}$$

and $|r_{\alpha,\beta}| < \infty$, $\alpha,\beta=1,2,\ldots,n$. In particular, if all m_α, $\alpha=1,2,\ldots,n$, are even, the condition that

$$\lim_{N\to\infty} \frac{1}{2\pi N} \int_{-\pi}^{\pi} K_N^2(\lambda) f(\lambda)\, d\lambda$$

exists and is finite is not needed.

The first thing we have to do in the proof of Theorem 3 is to justify the remark (ii) after Theorem 2. We first look at the subindex table, now it becomes

$$\begin{array}{llll} G_1 : & t_1+k_1^{(\alpha_1)} \ t_1+k_2^{(\alpha_1)} & \ldots & t_1+k_{m_{\alpha_1}}^{(\alpha_1)} \\ G_2 : & t_2+k_1^{(\alpha_2)} \ t_2+k_2^{(\alpha_2)} & \ldots & t_2+k_{m_{\alpha_2}}^{(\alpha_2)} \\ \cdots & \cdots & & \\ G_\ell : & t_\ell+k_1^{(\alpha_\ell)} \ t_\ell+k_2^{(\alpha_\ell)} & \ldots & t_\ell+k_{m_{\alpha_\ell}}^{(\alpha_\ell)} \end{array} \tag{30}$$

Let $G = G_1 \cup G_2 \cup \ldots \cup G_\ell$, so G contains $M_\ell = m_{\alpha_1} + m_{\alpha_2} + \ldots + m_{\alpha_\ell}$ elements.

Let $B_j = G - G_j$, $j=1,2,\ldots,\ell$.

We define a class of subsets of G, J to be such that

$$J = \{G_j ; j=1,2,\ldots,\ell\} \cup \{B_j : j=1,2,\ldots,\ell\}$$

Also, we write

$$\begin{cases} t_1+k_1^{(\alpha_1)} = s_1 \\ \cdots\cdots \\ t_1+k_{m_{\alpha_1}}^{(\alpha_1)} = s_{m_{\alpha_1}} \\ \cdots\cdots \\ \cdots\cdots \\ t_\ell+k_{m_{\alpha_\ell}}^{(\alpha_\ell)} = s_{M_\ell} \end{cases} \tag{31}$$

so we can identify each s_j with exactly one element in table (30) by (31).

We say

$$S = \{(s_{j_1}, s_{j_2}), (s_{j_3}, s_{j_4}), \ldots, (s_{j_{M_\ell - 1}}, s_{j_{M_\ell}})\}$$

is a <u>subindex-pair set</u> if $\{s_{j_1}, s_{j_2}, \ldots, s_{j_{M_\ell}}\}$ is a certain permutation of $s_1, s_2, \ldots, s_{M_\ell}$.

We say a subindex-pair set S is <u>decomposable with respect to J</u> if there are disjoint subsets S_1, S_2 of S such that $S = S_1 \cup S_2$ and $\cup S_1 = G_j, \cup S_2 = B_j$ for some $j = 1, 2, \ldots, \ell$. Otherwise S is <u>not decomposable with respect to J</u>.

We need a lemma to justify the remark (ii) after Theorem 2.

<u>Lemma 3.1</u> In the expansion of

$$E\left\{\left(X_{t_1+k_1}(\alpha_1) X_{t_1+k_2}(\alpha_1) \cdots X_{t_1+k_{m_{\alpha_1}}}(\alpha_1) - EX_{t_1+k_1}(\alpha_1) \cdots X_{t_1+k_{m_{\alpha_1}}}(\alpha_1)\right)\right.$$
$$\left(X_{t_2+k_1}(\alpha_2) X_{t_2+k_2}(\alpha_2) \cdots X_{t_2+k_{m_{\alpha_2}}}(\alpha_2) - EX_{t_2+k_1}(\alpha_2) \cdots X_{t_2+k_{m_{\alpha_2}}}(\alpha_2)\right)$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\left.\left(X_{t_\ell+k_1}(\alpha_\ell) X_{t_\ell+k_2}(\alpha_\ell) \cdots X_{t_\ell+k_{m_{\alpha_\ell}}}(\alpha_\ell) - EX_{t_\ell+k_1}(\alpha_\ell) \cdots X_{t_\ell+k_{m_{\alpha_\ell}}}(\alpha_\ell)\right)\right\}$$

the sum of all the terms whose subindex-pair sets are decomposable with respect to J is zero, i.e. after cancellation of the terms in the expansion, there is no term containing a factor like

$$E(X_{t_j+k_1}^{(\alpha_j)} X_{t_j+k_2}^{(\alpha_j)} \cdots X_{t_j+k_{m_{\alpha_j}}}^{(\alpha_j)})$$

for some $j=1,2,\ldots,\ell$.

[Proof] Write

$$W_j = X_{t_j+k_1}^{(\alpha_j)} X_{t_j+k_2}^{(\alpha_j)} \cdots X_{t_j+k_{m_{\alpha_j}}}^{(\alpha_j)}$$

$j=1,2,\ldots,\ell$.

Consider

$$E\{(W_1-EW_1)(W_2-EW_2)\ldots(W_\ell-EW_\ell)\} \tag{32}$$

Let A = sum of all the terms in the expansion of (32) whose subindex-pair set is decomposable with respect to J.

Let A_1 = sum of all the terms in the expansion of (32) which contains EW_1 as a factor.

A_2 = sum of all the terms in the expansion of (32) which contain EW_2, but not EW_1, as a factor.

. . . . .

A_ℓ = sum of all the terms in the expansion of (32) which contain EW_ℓ, but not EW_1 or EW_2 ... or $EW_{\ell-1}$, as a factor.

Then $A = A_1+A_2+\ldots+A_\ell$.

However

$$A_1 = E(W_1-EW_1)E\{(W_2-EW_2)(W_3-EW_3)\ldots(W_\ell-EW_\ell)\}$$

$$= 0$$

$$A_2 = E(W_2-EW_2)E\{(W_1-EW_1)(W_3-EW_3)\ldots(W_\ell-EW_\ell)\}$$
$$-E(W_2-EW_2)E(W_1-EW_1)E\{(W_3-EW_3)\ldots(W_\ell-EW_\ell)\}$$
$$= 0$$

$$A_3 = E(W_3-EW_3)E\{(W_1-EW_1)(W_2-EW_2)(W_4-EW_4)\ldots(W_\ell-EW_\ell)\}$$
$$-E(W_3-EW_3)E(W_1-EW_1)E\{(W_2-EW_2)(W_4-EW_4)\ldots(W_\ell-EW_\ell)\}$$
$$-E(W_3-EW_3)E(W_2-EW_2)E\{(W_1-EW_1)(W_4-EW_4)\ldots(W_\ell-EW_\ell)\}$$
$$+E(W_3-EW_3)E(W_1-EW_1)E(W_2-EW_2)E\{(W_4-EW_4)\ldots(W_\ell-EW_\ell)\}$$
$$= 0$$
$$\cdots\cdots\cdots\cdots$$
$$A_\ell = 0$$

Hence $A = 0$, Lemma 3.1 is proved.

Lemma 3.1 justifies Remark (ii) after Theorem 2, hence the proof of Theorem 3 goes exactly like that of Theorem 2. We shall state without proof several lemmas, and the conclusion of Theorem 3 follows immediately.

Lemma 3.2.

$$E\{Y_{N,\alpha}Y_{N,\beta}\} \longrightarrow r_{\alpha,\beta}$$

as $N \to \infty$, $\alpha,\beta=1,2,\ldots,n$ and moreover $|r_{\alpha,\beta}| < \infty$.
Note when all m_α are even, the condition (29) is not needed because no such term will appear in the proof of lemma 3.1.

Lemma 3.3. Let $A = \{\alpha_1,\alpha_2,\ldots,\alpha_\ell\}$, then

$$E\{Y_{N,\alpha_1} Y_{N,\alpha_2} \cdots Y_{N,\alpha_\ell}\}$$

$$\longrightarrow \begin{cases} 0 & \text{if } \ell \text{ is odd} \quad \text{(a)} \\ \tilde{\sum\limits_A} r_{\alpha_{j_1},\alpha_{j_2}} r_{\alpha_{j_3},\alpha_{j_4}} \cdots r_{\alpha_{j_{\ell-1}},\alpha_{j_\ell}} & \text{(b)} \\ & \text{if } \ell \text{ is even} \end{cases}$$

as $N \to \infty$, where $\tilde{\sum\limits_A}$ was defined in Theorem 1. Note in part (a) of lemma 3.3, we have $\to 0$ instead of $\equiv 0$ in lemma 2.6 (a).

Lemma 3.4

$$E\left(\sum_{\alpha=1}^{n} \mu_\alpha Y_{N,\alpha}\right)^{2\ell} \longrightarrow \frac{(2\ell)!}{2^\ell \ell!}\left[\sum_{\alpha_1=1}^{n} \sum_{\alpha_2=1}^{n} \mu_{\alpha_1}\mu_{\alpha_2} r_{\alpha_1,\alpha_2}\right]^\ell$$

$$E\left(\sum_{\alpha=1}^{n} \mu_\alpha Y_{N,\alpha}\right)^{2\ell+1} \longrightarrow 0$$

as $N \to \infty$ where μ_α, $\alpha=1,2,\ldots,n$ are arbitrary real numbers and $\ell=0,1,2,\ldots$.

4. A Further Generalization.

Let

$$W(t,k_1,\dots,k_m) = X_{t+k_1}X_{t+k_2}\dots X_{t+k_m}$$

$$Y_{M,\alpha}(t) = \sum_{m=1}^{M} \sum_{\substack{k_1,\dots,k_m=-\alpha \\ k_{i-1}\le k_i}}^{\alpha} a_{k_1,k_2,\dots,k_m} [W(t,k_1,\dots,k_m)-EW(t,k_1,\dots,k_m)]$$

$$W_N(k_1,k_2,\dots,k_m) = \frac{1}{N^{\frac{1}{2}}} \sum_{t=1}^{N} [W(t,k_1,\dots,k_m)-EW(t,k_1,\dots,k_m)]$$

$$Y_{N,M,\alpha} = \frac{1}{N^{\frac{1}{2}}} \sum_{t=1}^{N} Y_{M,\alpha}(t)$$

$$= \sum_{m=1}^{M} \sum_{\substack{k_1,\dots,k_m=-\alpha \\ k_{i-1}\le k_i}}^{\alpha} a_{k_1,k_2,\dots,k_m} W_N(k_1,k_2,\dots,k_m) \tag{33}$$

where $a_{k_1,k_2,\dots,k_m}$ are real numbers, and

$$C(k_1,k_2,\dots,k_m,k_1',k_2',\dots,k_{m'}')$$

$$= E[W(t,k_1,\dots,k_m)-EW(t,k_1,\dots,k_m)]$$

$$[W(t,k_1',\dots,k_{m'}')-EW(t,k_1',\dots,k_{m'}')]$$

$$r(k_1,k_2,\dots,k_m,k_1',k_2',\dots,k_{m'}')$$

$$= \lim_{N\to\infty} EW_N(k_1,k_2,\dots,k_m)W_N(k_1',k_2',\dots,k_{m'}') \tag{34}$$

Lemma 3.2 assures the existence of the limit r.

Theorem 4.1. Suppose

$$\sum_{m=1}^{\infty} \sum_{\substack{k_1=-\infty \\ i=1,\dots,m \\ k_{i-1}\le k_i}}^{\infty} \sum_{m'=1}^{\infty} \sum_{\substack{k_1'=-\infty \\ i=1,\dots,m' \\ k_{i-1}'\le k_i'}}^{\infty} |a_{k_1,\dots,k_m} a_{k_1',\dots,k_{m'}'} c(k_1,\dots,k_m,k_1',\dots,k_{m'}')| < \infty \tag{35}$$

Then

$$Y_{M,\alpha}(t) \xrightarrow{\text{in the mean}} Y(t) = \sum_{m=1}^{\infty} \sum_{\substack{k_i=-\infty \\ i=1,\ldots,m \\ k_{i-1}\leq k_i}}^{\infty} a_{k_1,k_2,\ldots,k_m} [W(t,k_1,\ldots,k_m)-EW(t,k_1,\ldots,k_m)] \tag{36}$$

as $M,\alpha \rightarrow \infty$, where

$$E|Y(t)|^2 = \lim_{M,\alpha\rightarrow\infty} E|Y_{M,\alpha}(t)|^2$$

$$= \sum_{m=1}^{\infty} \sum_{\substack{k_i=-\infty \\ i=1,\ldots,m \\ k_{i-1}\leq k_i}}^{\infty} \sum_{m'=1}^{\infty} \sum_{\substack{k_i'=-\infty \\ i=1,\ldots,m' \\ k_{i-1}'\leq k_i'}}^{\infty} a_{k_1,k_2,\ldots,k_m} a_{k_1',k_2',\ldots,k_{m'}'} c(k_1,k_2,\ldots,k_m,k_1',k_2',\ldots,k_{m'}') \tag{37}$$

[Proof] Let $M_1,M_2,\alpha_1,\alpha_2$ be any positive integers and let

$$M = \min(M_1,M_2)$$
$$\alpha = \min(\alpha_1,\alpha_2)$$

Then

$$E|Y_{M_1,\alpha_1}(t) - Y_{M_2,\alpha_2}(t)|^2$$

$$= \left(\sum_{m=1}^{M_1} \sum_{\substack{k_1,\ldots,k_m=-\alpha_1 \\ k_{i-1}\leq k_i}}^{\alpha_1} - \sum_{m=1}^{M_2} \sum_{\substack{k_1,\ldots,k_m=-\alpha_2 \\ k_{i-1}\leq k_i}}^{\alpha_2} \right) \left(\sum_{m'=1}^{M_1} \sum_{\substack{k_1',\ldots,k_m'=-\alpha_1 \\ k_{i-1}'\leq k_i'}}^{\alpha_1} - \sum_{m'=1}^{M_2} \sum_{\substack{k_1',\ldots,k_m'=-\alpha_2 \\ k_{i-1}'\leq k_i'}}^{\alpha_2} \right) a_{k_1,\ldots,k_m} a_{k_1',\ldots,k_{m'}'} c(k_1,\ldots,k_m,k_1',\ldots,k_{m'}')$$

$$\leq \left(\sum_{m=1}^{\infty} \sum_{\substack{\alpha<|k_i| \\ i=1,\ldots,m \\ k_{i-1}\leq k_i}} + \sum_{M<m} \sum_{\substack{k_i=-\infty \\ i=1,\ldots,m \\ k_{i-1}\leq k_i}}^{\infty} \right) \left(\sum_{m'=1}^{\infty} \sum_{\substack{\alpha<|k_i'| \\ i=1,\ldots,m' \\ k_{i-1}'\leq k_i'}} + \sum_{M<m'} \sum_{\substack{k_i'=-\infty \\ i=1,\ldots,m' \\ k_{i-1}'\leq k_i'}}^{\infty} \right)$$

$$|a_{k_1,\ldots,k_m} a_{k_1',\ldots,k_{m'}'} c(k_1,\ldots,k_m,k_1',\ldots,k_{m'}')| \qquad (38)$$

By (35), for each $\varepsilon > 0$, we can choose an $M(\varepsilon) > 0$ and an $\alpha(\varepsilon) > 0$ such that

$$\sum_{m=1}^{\infty} \sum_{\substack{\alpha<|k_i| \\ i=1,\ldots,m \\ k_{i-1}\leq k_i}} \sum_{m'=1}^{\infty} \sum_{\substack{k_i'=-\infty \\ i=1,\ldots,m' \\ k_{i-1}'\leq k_i'}}^{\infty} |a_{k_1,\ldots,k_m} a_{k_1',\ldots,k_{m'}'} c(k_1,\ldots,k_m,k_1',\ldots,k_{m'}')| < \frac{\varepsilon}{4}$$

and

$$\sum_{M<m} \sum_{\substack{k_i=-\infty \\ i=1,\ldots,m \\ k_{i-1}\leq k_i}}^{\infty} \sum_{m'=1}^{\infty} \sum_{\substack{k_i'=-\infty \\ i=1,\ldots,m' \\ k_{i-1}\leq k_i}}^{\infty} |a_{k_1,\ldots,k_m} a_{k_1',\ldots,k_{m'}'} c(k_1,\ldots,k_m,k_1',\ldots,k_{m'}')| < \frac{\varepsilon}{4} \qquad (39)$$

whenever $M > M(\varepsilon)$ and $\alpha > \alpha(\varepsilon)$.

From (38) and (39), it follows

$$E|Y_{M_1,\alpha_1}(t) - Y_{M_2,\alpha_2}(t)|^2 < \varepsilon$$

whenever $M_1, M_2 > M(\varepsilon)$ and $\alpha_1, \alpha_2 > \alpha(\varepsilon)$. Hence $Y_{M,\alpha}(t)$ converges in the mean as $M, \alpha \to \infty$ and

$$Y_{M,\alpha}(t) \xrightarrow{\text{in the mean}} Y(t)$$

$$= \sum_{m=1}^{\infty} \sum_{\substack{k_1,\ldots,k_m=-\infty \\ k_{i-1}\leq k_i}}^{\infty} a_{k_1,\ldots,k_m} [W(t,k_1,\ldots,k_m) - EW(t,k_1,\ldots,k_m)]$$

The assertion (37) follows easily from (35). Q.E.D.

We have shown that given (35) the $Y(t)$ in (36) exists in the mean. Now we should like to find a condition on the coefficients $a_{k_1,k_2,\ldots,k_m}$ such that

$$Y_N = \frac{1}{N^{\frac{1}{2}}} \sum_{t=1}^{N} Y(t)$$

is asymptotically normally distributed.

Let

$$r_N(k_1,k_2,\ldots,k_m,k_1',k_2',\ldots,k_{m'}')$$
$$= EW_N(k_1,k_2,\ldots,k_m)W_N(k_1',k_2',\ldots,k_{m'}')$$

and

$$\eta(k_1,\ldots,k_m,k_1',\ldots,k_{m'}') = \sup_{1\leq N<\infty} r_N(k_1,\ldots,k_m,k_1',\ldots,k_{m'}') \tag{40}$$

Then one of the sufficient conditions for Y_N asymptotically normally distributed is

$$\sum_{m=1}^{\infty} \sum_{\substack{k_1,\ldots,k_m=-\infty \\ k_{i-1}\leq k_i}}^{\infty} \sum_{m'=1}^{\infty} \sum_{\substack{k_1',\ldots,k_{m'}'=-\infty \\ k_{i-1}'\leq k_i'}}^{\infty} |a_{k_1,\ldots,k_m} a_{k_1',\ldots,k_{m'}'} \eta(k_1,k_2,\ldots,k_m,k_1',k_2',\ldots,k_{m'}')| < \infty \tag{41}$$

Condition (41) assures the uniform convergence of

$$\beta_N = \sum_{m=1}^{\infty} \sum_{\substack{k_i=-\infty \\ i=1,\ldots,m \\ k_{i-1}\leq k_i}}^{\infty} \sum_{m'=1}^{\infty} \sum_{\substack{k_i'=-\infty \\ i=1,\ldots,m' \\ k_{i-1}'\leq k_i'}}^{\infty} |a_{k_1,\ldots,k_m} a_{k_1',\ldots,k_{m'}'} r_N(k_1,\ldots,k_m,k_1',\ldots,k_{m'}')| \tag{42}$$

for all N, which is sufficient to give asymptotic normality of Y_N. The proof will be carried out in Theorem 4.2.

(41) is not a very unpleasant condition, since we can easily give an upper bound for each $\eta(k_1,\ldots,k_m,k_1',\ldots,k_{m'}')$ independent of N. By Corollary 1.1

$$EW_N(k_1,\ldots,k_m)W_N(k_1',\ldots,k_{m'}')$$

$$= \frac{1}{N}\sum_{t_1=1}^{N}\sum_{t_2=1}^{N} E[X_{t_1+k_1}\ldots X_{t_1+k_m} - E(X_{t_1+k_1}\ldots X_{t_1+k_m})]$$
$$[X_{t_2+k_1'}\ldots X_{t_2+k_{m'}'} - E(X_{t_2+k_1'}\ldots X_{t_2+k_{m'}'})]$$

$$\begin{cases} = 0 & \text{if } m+m' \text{ is odd} \\ = \frac{1}{N}\sum_{t_1=1}^{N}\sum_{t_2=1}^{N}\sum^{*}_{A_m\cup A_{m'}} r_{k_{j_1}-k_{j_2}} r_{k_{j_3}-k_{j_4}}\cdots r_{k_{j_{2\omega-1}}-k_{j_{2\omega}}} & \\ \quad r_{k'_{p_1}-k'_{p_2}} r_{k'_{p_3}-k'_{p_4}}\cdots r_{k'_{p_{2\omega+2\beta-1}}-k'_{p_{2\omega+2\beta}}} & \\ \quad r_{(t_1+k_{j_{2\omega+1}}-t_2-k'_{p_{2\omega+2\beta+1}})}\cdots r_{(t_1+k_{j_m}-t_2+k'_{p_{m'}})} & \\ & \text{if } m+m' \text{ is even} \end{cases}$$

where $A_m = \{k_1,\ldots,k_m\}$, $A_{m'} = \{k_1',\ldots,k_{m'}'\}$ and $\beta = \frac{m'-m}{2}$.

By a simple estimation,

$$EW_N(k_1,\ldots,k_m)W_N(k_1',\ldots,k_{m'}') \equiv 0 \quad \text{if } m+m' \text{ is odd}$$

and $$|EW_N(k_1,\ldots,k_m)W_N(k_1',\ldots,k_{m'}')|$$

$$\leq \frac{(m+m')!}{2^{\frac{m+m'}{2}}\left(\frac{m+m'}{2}\right)!}\,\bar{r}^{\frac{m+m'}{2}}\,C \qquad \text{if } m+m' \text{ is even} \tag{43}$$

for all N, where

$$\bar{r} = \max\left(\max_{-\infty < t < \infty} r_t, 1\right)$$

$$C = \max\left(\sum_{t=-\infty}^{\infty} r_t^2, \ \sup_N \frac{1}{N} \int_{-\infty}^{\infty} K_N^2(\lambda) f(\lambda) d\lambda \right) \tag{44}$$

Hence, we can replace $\eta(k_1,\ldots,k_m,k_1',\ldots,k_{m'}')$ in (41) by

$$\bar{\eta}(k_1,\ldots,k_m,k_1',\ldots,k_{m'}') \begin{cases} = 0 & \text{if } m+m' \text{ is odd} \\ = \dfrac{(m+m')!}{2^{\frac{m+m'}{2}} \left(\frac{m+m'}{2}\right)!} \bar{r}^{\frac{m+m'}{2}} & \text{if } m+m' \text{ is even} \end{cases} \tag{45}$$

and get a more explicit sufficient condition for asymptotic normality of $\frac{1}{N^{\frac{1}{2}}} \sum_{t=1}^{N} Y(t)$

$$\sum_{m=1}^{\infty} \sum_{\substack{k_1,\ldots,k_m=-\infty \\ k_{i-1}\le k_i}}^{\infty} \sum_{m'=1}^{\infty} \sum_{\substack{k_1',\ldots,k_{m'}'=-\infty \\ k_{i-1}'\le k_i'}}^{\infty} \left| a_{k_1,\ldots,k_m} a_{k_1',\ldots,k_{m'}'} \right| \bar{\eta}(k_1,\ldots,k_m,k_1',\ldots,k_m') < \infty \tag{46}$$

<u>Theorem 4.2</u> Suppose $f \in L^2(-\pi,\pi)$ and

$$\lim_{N\to\infty} \frac{1}{2\pi N} \int_{-\pi}^{\pi} K_N^2(\lambda) f(\lambda) d\lambda$$

exists and is finite. Then

$$Y_N = \frac{1}{N^{\frac{1}{2}}} \sum_{t=1}^{N} Y(t)$$

where $Y(t)$ was defined in (36), is asymptotically normally distributed with mean zero and variance

$$\rho = \sum_{m=1}^{\infty} \sum_{\substack{k_1,\ldots,k_m=-\infty \\ k_{i-1}\le k_i}}^{\infty} \sum_{m'=1}^{\infty} \sum_{\substack{k_1',\ldots,k_{m'}'=-\infty \\ k_{i-1}'\le k_i'}}^{\infty} a_{k_1,\ldots,k_m} a_{k_1',\ldots,k_{m'}'} \, r(k_1,\ldots,k_m,k_1',\ldots,k_{m'}') \tag{47}$$

where $r(k_1,\dots,k_m,k_1',\dots,k_{m'}')$ is defined in (34), if (41) holds or in particular if (46) holds.

The condition $\lim_{N\to\infty} \frac{1}{2\pi N} \int_{-\infty}^{\infty} K_N^2(\lambda) f(\lambda) d\lambda$ exists and is finite is not needed if $a_{k_1,k_2,\dots,k_m}=0$ whenever m is odd.

[Proof] As we have remarked before condition (41) implies that

$$\beta_N = \sum_{m=1}^{\infty} \sum_{\substack{k_i=-\infty \\ i=1,\dots,m \\ k_{i-1}\le k_i}}^{\infty} \sum_{m'=1}^{\infty} \sum_{\substack{k_i'=-\infty \\ i=1,\dots,m' \\ k_{i-1}'\le k_i'}}^{\infty} |a_{k_1,\dots,k_m} a_{k_1',\dots,k_{m'}'} r_N(k_1,\dots,k_m,k_1',\dots,k_{m'}')|$$

converges uniformly for all N and β_N is a uniformly bounded sequence. Therefore

$$Y_{N,M,\alpha} = \sum_{m=1}^{M} \sum_{\substack{k_1,\dots,k_m=-\alpha \\ k_{i-1}\le k_i}}^{\alpha} a_{k_1,k_2,\dots,k_m} W_N(k_1,k_2,\dots,k_m)$$

$$\xrightarrow{\text{in the mean}} Y_N = \frac{1}{N^{\frac{1}{2}}} \sum_{t=1}^{N} Y(t)$$

$$= \sum_{m=1}^{\infty} \sum_{\substack{k_1,\dots,k_m=-\infty \\ k_{i-1}\le k_i}}^{\infty} a_{k_1,\dots,k_m} W_N(k_1,\dots,k_m) \quad (48)$$

uniformly for all N, as $M,\alpha \longrightarrow \infty$.

Let

$$\rho_{M,\alpha} = \lim_{N\to\infty} E|Y_{N,M,\alpha}|^2$$

$\rho_{M,\alpha}$ exists by Theorem 3 and

$\rho_{M,\alpha}$

$$= \sum_{m=1}^{M} \sum_{\substack{|k_i|\le\alpha \\ i=1,\ldots,m \\ k_{i-1}\le k_i}} \sum_{m'=1}^{M} \sum_{\substack{|k_i'|\le\alpha \\ i=1,\ldots,m' \\ k_{i-1}'\le k_i'}} a_{k_1,\ldots,k_m} a_{k_1',\ldots,k_{m'}'} r(k_1,\ldots,k_m,k_1',\ldots,k_{m'}') .$$

By (41) or (46), we have the ρ of (47) exists and is finite and

$$\rho_{M,\alpha} \longrightarrow \rho \qquad \text{as } M,\alpha \longrightarrow \infty \tag{49}$$

It is easy to show that (48) implies, for fixed t,

$$E\{e^{itY_{N,M,\alpha}}\} \longrightarrow E\{e^{itY_N}\} \tag{50}$$

uniformly for all N as $M,\alpha \longrightarrow \infty$. By (49) and (50) we have, for each $\varepsilon > 0$ and fixed t, an $M(\varepsilon,t) > 0$ and an $\alpha(\varepsilon,t) > 0$ such that

$$|e^{-\frac{1}{2}\rho_{M,\alpha}t^2} - e^{-\frac{1}{2}\rho t^2}| < \frac{\varepsilon}{3} \tag{51}$$

and

$$|E\{e^{itY_{N,M,\alpha}}\} - E\{e^{itY_N}\}| < \frac{\varepsilon}{3} \tag{52}$$

whenever $M > M(\varepsilon,t)$ and $\alpha > \alpha(\varepsilon,t)$. By Theorem 3, for each (ε,M,α,t), we have an $N(\varepsilon,M,\alpha,t) > 0$ such that

$$\left|E\{e^{itY_{N,M,\alpha}}\} - e^{-\frac{1}{2}\rho_{M,\alpha}t^2}\right| < \frac{\varepsilon}{3} \tag{53}$$

whenever $N > N(\varepsilon,M,\alpha,t)$. By (51), (52) and (53), we have, for each t,

$$
\begin{aligned}
&|E\{e^{itY_N}\} - e^{-\frac{1}{2}\rho t^2}| \\
&\leq |E\{e^{itY_N}\} - E\{e^{itY_{N,M,\alpha}}\}| \\
&+ |E\{e^{itY_{N,M,\alpha}}\} - e^{-\frac{1}{2}\rho_{M,\alpha} t^2}| \\
&+ |e^{-\frac{1}{2}\rho_{M,\alpha} t^2} - e^{-\frac{1}{2}\rho t^2}| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon
\end{aligned}
$$

whenever $N > N(\varepsilon, M, \alpha, t)$. Q.E.D.

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